Recursive Functional Link Polynomial Filters: an Introduction

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***Abstract*—In this paper, we first introduce a novel sub-class of recursive linear-in-the-parameters nonlinear filters, called recursive functional link polynomial filters, which are derived by using the constructive rule of Volterra filters. These filters are universal approximators, according to the Stone-Weierstrass theorem, and offer a remedy to the main drawback of their finite memory counterparts, that is the curse of dimensionality. Since recursive nonlinear filters become, in general, unstable for large input signals, we then consider a simple stabilization procedure by slightly modifying the input-output relationship of recursive functional link polynomial filters. The resulting filters are always stable and, even though no more universal approximators, still offer good modeling performance for nonlinear systems.**

***Index Terms*—Linear-in-the-parameters nonlinear filters, recursive functional link polynomial filters, universal approximators, bounded-input bounded-output stability.**

I. Introduction

Among the variety of nonlinear models studied in the literature, linear-in-the-parameters (LIP) nonlinear filters are widely utilized for the identification of unknown nonlinear systems. Their characterizing property is the linearity of the output with respect to the filter coefficients. Applications range from the computation of efficient models [1], [2], [3], [4], [5], to nonlinear active noise control [6], [7], [8], [9] and nonlinear acoustic echo cancellation [10], [11], [12], [13], [14].

LIP nonlinear filters can be subdivided in models with finite or infinite memory [15]. Recently, a sub-class of the finite-memory LIP nonlinear filters formed with the so-called functional link polynomial (FLiP) filters, has been introduced [16]. The basis functions of FLiP filters are polynomials of nonlinear expansions of delayed input samples. The characterizing property of these filters is the fact that, by construction, their basis functions satisfy the same conditions of the triangular representation of Volterra filters. Therefore, all the filters in this sub-class are universal approximators, according to the Stone-Weierstrass theorem [17].

Nonlinear filters with infinite memory have also been introduced, as recursive second-order Volterra filters [18], [19], [20], [21] recursive functional link artificial neural networks [22], [23] and bilinear filters [15]. These filters are usually more parsimonious in the use of coefficients than their finite memory counterparts, and thus constitute an efficient solution to limit the complexity of finite memory models. The drawback is the fact that they usually become unstable for large amplitudes of the input signal. In the literature, this problem has been solved by deriving sufficient conditions for the bounded-input bounded-output (BIBO) stability, which are specific for any filter.

In this paper, we first introduce the infinite-memory versions of the FLiP filters mentioned above, using a recursive input-output relationship that exploits finite sets of input and past output samples. We show that, within simple conditions, recursive FLiP (RFLiP) filters satisfy the Stone-Weierstrass theorem. Therefore, RFLiP filters are able to arbitrarily well approximate any causal, time-invariant, infinite-memory, continuous, nonlinear system. Then, we consider the problem of their stability according to the BIBO criterion. We slightly modify the input-output relationship of the RFLiP filters using a simple nonlinear mapping of the basis functions depending on the past output samples. As a consequence, the BIBO stability is guaranteed for any finite-amplitude input signal. The stabilized RFLiP (SRFLiP) filters are no more, in principle, universal approximators but give anyway sufficiently good approximation performance in real-world environments.

The paper is organized as follows. In Section II, RFLiP filters are introduced as universal approximators for recursive nonlinear systems. Stabilized RFLiP filters are considered in Section III. Validation results, including an experiment on a benchmark for nonlinear system identification, are presented in Section IV. Conclusions follow in Section V.

II. Infinite-memory functional link polynomial filters

We assume here that the unknown nonlinear system with infinite memory is represented by the input-output relationship

$$y(n) = f[x(n), \dots, x(n-N), y(n-1), \dots, y(n-M)], \quad (1)$$

where f is a real continuous function and $x(n)$, $y(n)$ are real-valued sequences. In other words, at each time step n, the output is considered as a function of the present and N past values of the input $x(n)$ and of the M past values of the output $y(n)$. Then, the input-output relationship of an RFLiP filter is given by

$$\hat{y}(n) = \hat{f}[\xi_0, \dots, \xi_N, \xi_{N+1}, \dots, \xi_{N+M}], \quad (2)$$

where, for convenience,

$$\xi_0 = x(n), \dots, \xi_N = x(n-N),$$
$$\xi_{N+1} = \hat{y}(n-1), \dots, \xi_{N+M} = \hat{y}(n-M), \quad (3)$$

and $\hat{y}(n)$ is the output of the RFLiP filter. In (2), $\hat{f}$ is assumed to be a linear combination of basis functions $f_i(n)$ in the variables $\xi_0, \ldots, \xi_N, \xi_{N+1}, \ldots, \xi_{N+M}$,

$$\hat{y}(n) = \sum_{i=0}^{N_{\mathrm{T}}-1} h_i f_i(n). \tag{4}$$

with N_{T} the total number of basis functions considered. The derivation of a complete set of basis functions f_i for the RFLiP filter follows the same rule used for FLiP filters in [16]. Let us assume that a set of basis functions without memory satisfying all the requirements of the Stone-Weierstrass theorem is given by the ordered set

$$g_0[\xi], g_1[\xi], g_2[\xi], g_3[\xi], \ldots \tag{5}$$

where $g_0[\xi]$ is the basis function of order 0, usually assumed equal to 1, $g_1[\xi]$ is an odd basis function of order 1, $g_2[\xi]$ is an even basis function of order 2, and so on. A complete set of basis functions for a system with memory is obtained by writing first the basis functions $g_j[\xi]$ in (5) for the arguments $\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}, \ldots, \xi_{N+M}$. Then, the functions g_j of different variables are multiplied one to each other in any possible manner, taking care of avoiding repetitions, as in the triangular representation of Volterra filters. It should be noted that in such a way cross products of basis functions involving input and past output samples are formed, in contrast to polynomial filters previously introduced in the literature [18], [19], [20], [21]. The basis functions f_i for an RFLiP filter of order $P = 0, 1, 2, 3$ are given in Table I. It is worth noting that the total number N_T of basis functions, or filter coefficients, including all the kernels of order $0, \ldots, P$, is equal to that of a Volterra filter of the same order and memory equal to $N + M + 1$,

$$N_{\mathrm{T}} = \binom{N+M+1+P}{N+M+1}. \tag{6}$$

Even though the number of terms increases geometrically with the parameters N, M and exponentially with the order P, RFLiP filters are often able to provide much more compact representations than their non-recursive counterparts. Therefore, they constitute an efficient solution to limit the complexity of the nonlinear models.

TABLE I
BASIS FUNCTIONS f_i OF THE RFLIP FILTER

Order 0
$g_0 = 1.$
Order 1
$g_1[\xi_0], \ldots, g_1[\xi_N], g_1[\xi_{N+1}], \ldots, g_1[\xi_{N+M}].$
Order 2
$g_2[\xi_0], \ldots, g_2[\xi_N], g_2[\xi_{N+1}], \ldots, g_2[\xi_{N+M}],$ $g_1[\xi_0]g_1[\xi_1], \ldots, g_1[\xi_{N+M-1}]g_1[\xi_{N+M}],$ $g_1[\xi_0]g_1[\xi_2], \ldots, g_1[\xi_{N+M-2}]g_1[\xi_{N+M}],$ $\vdots$ $g_1[\xi_0]g_1[\xi_{N+M}].$
Order 3
$g_3[\xi_0], \ldots, g_3[\xi_N], g_3[\xi_{N+1}], \ldots, g_3[\xi_{N+M}],$ $g_2[\xi_0]g_1[\xi_1], \ldots, g_2[\xi_{N+M-1}]g_1[\xi_{N+M}],$ $\vdots$ $g_2[\xi_0]g_1[\xi_{N+M}],$ $g_1[\xi_0]g_2[\xi_1], \ldots, g_1[\xi_{N+M-1}]g_2[\xi_{N+M}],$ $\vdots$ $g_1[\xi_0]g_2[\xi_{N+M}],$ $\vdots$ $g_1[\xi_0]g_1[\xi_1]g_1[\xi_2], \ldots,$ $g_1[\xi_{N+M-2}]g_1[\xi_{N+M-1}]g_1[\xi_{N+M}].$

A. RFLiP filters as universal approximators

Let us assume that the unknown system with the input-output relationship in (1) is stable. In what follows, we prove that RFLiP filters can arbitrarily well approximate the system in (1), and thus are universal approximators. The proof follows similar arguments as in [24]. The following two stability conditions are exploited.

Assumption 1: The system in (1) is BIBO stable for $|x(n)| \leq R$ with $|y(n)| \leq A$ for all n.

Without loss in generality, we can assume $R = 1$ and $A = 1$. In fact, if $R \neq 1$ and/or $A \neq 1$, $x(n)$ and $y(n)$ can be scaled as $x'(n) = x(n)/R$ and $y'(n) = y(n)/A$ and then the identification refers to the equivalent system

$$y'(n) = f'[x'(n), \ldots, x'(n-N), y'(n-1), \ldots, y'(n-M)] =$$
$$\frac{1}{A} f[Rx'(n), \ldots, Rx'(n-N), Ay'(n-1), \ldots, Ay'(n-M)].$$

Assumption 2: Given the perturbed system

$$\widetilde{y}(n) = f[x(n), \ldots, x(n-N), \widetilde{y}(n-1), \ldots, \widetilde{y}(n-M)] + \nu(n), \tag{7}$$

for any $\theta > 0$ there exist $\epsilon > 0$ such that when $|\nu(n)| < \epsilon \;\forall n$ it is

$$|y(n) - \widetilde{y}(n)| < \theta \quad \forall n, \tag{8}$$

with $y(n)$ the output of the system in (1).

Assumption 2 means that if we apply a small perturbation to the system in (1), the output remains close to $y(n)$.

If these conditions are satisfied, the recursive nonlinear system in (1) is arbitrarily well approximated by an appropriate set of basis functions built as those shown in Table I up to the order $P = 3$. Indeed, for any $\epsilon > 0$, according to the Stone-Weierstrass theorem, there is a linear combination of basis functions, shortly noted as $\widetilde{f}(\xi_0, \ldots, \xi_N, \xi_{N+1}, \ldots, \xi_{N+M})$, such that for any set of values $\xi_0, \ldots, \xi_N, \xi_{N+1}, \ldots, \xi_{N+M}$ in $[-1, +1]$ it results

$$|f(\xi_0, \ldots, \xi_N, \xi_{N+1}, \ldots, \xi_{N+M}) - \widetilde{f}(\xi_0, \ldots, \xi_N, \xi_{N+1}, \ldots, \xi_{N+M})| < \epsilon. \tag{9}$$

Let us now consider the system

$$\widetilde{y}(n) = \widetilde{f}[x(n), \ldots, x(n-N), \widetilde{y}(n-1), \ldots, \widetilde{y}(n-M)]. \tag{10}$$

According to (9) it is

$$\widetilde{y}(n) = f[x(n), \ldots, x(n-N), \widetilde{y}(n-1), \ldots, \widetilde{y}(n-M)] + \nu(n) \tag{11}$$

with $|\nu(n)| < \epsilon$. According to Assumption 2 and the Stone-Weierstrass theorem, for any $\theta > 0$ there is a sufficiently small $\epsilon > 0$ and a linear combination of basis functions such that the error between the output of (1) and (10) is $|y(n) - \widetilde{y}(n)| < \theta$. This fact means that, by using a sufficiently large number N_{T} of basis functions of a sufficiently high order P, it is possible to approximate the recursive system in (1) within any required precision. Therefore, the output $\hat{y}(n)$ of the modeling filter can be written as in (4).

B. Members of the RFLiP sub-class

Any choice of the set of 1-dimensional basis functions $g_j[\xi]$, satisfying the requirements of the Stone-Weierstrass theorem, leads to a different type of RFLiP filter. For example, a recursive Volterra filter is defined by using as $g_j[\xi]$ the monomials

$$1, \xi, \xi^2, \xi^3, \ldots \tag{12}$$

A recursive even mirror Fourier nonlinear (EMFN) filter is defined by using as $g_j[\xi]$ the even mirror symmetric trigonometric functions

$$1, \sin[\frac{1}{2}\pi\xi], \cos[\pi\xi], \sin[\frac{3}{2}\pi\xi], \ldots, \cos[l\pi\xi], \sin[\frac{2l+1}{2}\pi\xi], \ldots \tag{13}$$

where 1 is the basis function of order 0, $\sin[\frac{2l+1}{2}\pi\xi]$ is a basis function of order $2l+1$, and $\cos[l\pi\xi]$ is a basis function of order $2l$, with l a natural number [25].

A recursive Legendre nonlinear filter is defined by using as $g_j[\xi]$ the Legendre polynomials

$$\mathrm{leg}_0[\xi], \mathrm{leg}_1[\xi], \mathrm{leg}_2[\xi], \mathrm{leg}_3[\xi], \ldots \tag{14}$$

The Legendre polynomials $\mathrm{leg}_{l+1}[\xi]$ are obtained from the recursive relation

$$\mathrm{leg}_{l+1}[\xi] = \frac{2l+1}{l+1}\xi \cdot \mathrm{leg}_l[\xi] - \frac{l}{l+1}\mathrm{leg}_{l-1}[\xi], \tag{15}$$

where $\mathrm{leg}_0[\xi] = 1$, $\mathrm{leg}_1[\xi] = \xi$, and the natural number l is the order of the basis function [26].

A recursive Chebyshev nonlinear filter is defined by using as $g_j[\xi]$ the Chebyshev polynomials of first kind

$$\mathrm{T}_0[\xi], \mathrm{T}_1[\xi], \mathrm{T}_2[\xi], \mathrm{T}_3[\xi], \ldots \tag{16}$$

The Chebyshev polynomials are generated by the following recursive relation:

$$T_{l+1}[\xi] = 2\xi T_l[\xi] - T_{l-1}[\xi], \tag{17}$$

where $T_0[\xi] = 1$, $T_1[\xi] = \xi$ and the natural number l is the order of the basis function [27].

III. Stable recursive functional link polynomial filters

It is well known that recursive filters with fixed coefficients become unstable, according to the BIBO criterion, for given input signals. In contrast, it has been recently shown that the recursive EMFN filter, a member of the RFLiP sub-class, is intrinsically BIBO stable [24]. Unfortunately, the other members of this sub-class do not possess this property. Instead of deriving sufficient conditions that merely fix an upper bound on the amplitude of the input signal, our aim is here to stabilize these filters. To this purpose, we define a new sub-class of stable RFLiP (SRFLiP) filters which is obtained by using the hyperbolic tangent to bound to the unity the basis functions depending on the past outputs. More specifically, we replace the basis functions $g_j[\xi]$ of the output arguments $\xi_{N+1}, \ldots, \xi_{N+M}$ with the functions $\tanh\{g_j[\xi]\}$. As a consequence, any resulting basis function $\bar{f}_i(n)$ is bounded by an appropriate finite number $|k_i|$ for any input signal with finite amplitude. Then, from (4) it results

$$|\hat{y}(n)| \le \sum_{i=0}^{N_{\mathrm{T}}-1} |h_i||\bar{f}_i(n)| \le \sum_{i=0}^{N_{\mathrm{T}}-1} |h_i||k_i| = K, \tag{18}$$

where K is a finite number. Therefore, for any bounded input signal $x(n)$ the output signal $\hat{y}(n)$ is also bounded for any n. It should be noted that the SRFLiP filters are no more universal approximators. However, for small input signals the behavior of SRFLiP is close to that of RFLiP filters, and even for large input signals their approximation performance remain sufficiently good.

IV. Experimental results

A. The adaptation algorithm

To model an unknown system, adaptive RFLiP and SRFLiP filters can be exploited. In this section, we use as adaptation algorithm the normalized version of the output-error pseudo-LMS algorithm based on the pseudolinear regression proposed in [28] for adaptive IIR filters. Indeed, it is a simple algorithm that can be used for recursive LIP nonlinear filters, too [15]. Let us arrange the basis functions in (4) (or the modified basis function $\bar{f}_i(n)$) in the vector

$$\mathbf{f}(n) = [f_1(n), f_2(n), \ldots, f_{N_{\mathrm{T}}}(n)]^T \tag{19}$$

and the corresponding coefficients in the vector

$$\mathbf{h}(n) = [h_1(n), h_2(n), \ldots, h_{N_{\mathrm{T}}}(n)]^T. \tag{20}$$

Then, the pseudo-NLMS algorithm is given by

$$\mathbf{h}(n+1) = \mathbf{h}(n) + \frac{\mu}{\mathbf{f}^T(n)\mathbf{f}(n)}\mathbf{f}(n)e(n), \tag{21}$$

where μ is the step size and $e(n) = y(n) - \hat{y}(n)$. The complexity of the algorithm is of $\mathcal{O}(N_{\mathrm{T}})$ multiplications and additions per time instant.

B. An experiment on BIBO stability

We first propose a simulation experiment to check the stability of SRFLiP filters. The unknown system is described by a second-order REMFN filter which is BIBO stable for any input signal with finite amplitude

$$\begin{aligned} y(n) = 0.7733\sin[\pi/2 \cdot x(n)] - \\ 0.0991\sin[\pi/2 \cdot x(n-1)]\sin[\pi/2 \cdot x(n-2)] - \\ 0.3160\sin[\pi/2 \cdot y(n-1)] + 0.0052\cos[\pi y(n-1)] - \\ 0.3756\sin[\pi/2 \cdot y(n-1)]\sin[\pi/2 \cdot y(n-2)] + \\ 0.3500\sin[\pi/2 \cdot x(n-1)]\sin[\pi/2 \cdot y(n-2)]. \end{aligned} \tag{22}$$

TABLE II
COEFFICIENTS OF THE RV AND SRV FILTERS

RV	$a_1 = 1.2062$, $a_{12} = 0.3693$, $b_1 = -0.4917$, $b_{11} = 0.1586$, $b_{12} = -1.0153$, $c_{12} = 0.8122$
SRV	$a'_1 = 1.2061$, $a'_{12} = 0.3890$, $b'_1 = -0.4968$, $b'_{11} = 0.1715$, $b'_{12} = -1.0523$, $c'_{12} = 0.8310$

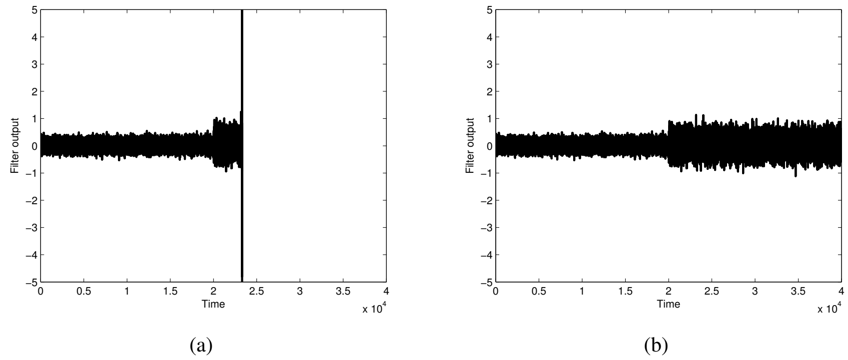

Fig. 1. Outputs from the modeling filters: (a) RV filter, (b) stabilized RV filter.

TABLE III
NMSE IN dB FOR THE SILVER BOX EXPERIMENT

REMFN	RV	SRV	RL	SRL	RC	SRC	BIL	SBIL	EMFN	VOL	LEG	CHEB
-18.2	-19.7	-19.6	-19.2	-19.0	-18.6	-17.9	-18.6	-18.6	-3.9	-4.0	-3.9	-3.9

In our simulations, we use as input a zero-mean Gaussian signal with variance 0.01, so that the range of the input signal is about ± 0.42. The unknown system is modeled with a second-order recursive Volterra (RV) filter and a stabilized RV (SRV) filter, respectively. The RV and SRV filters have the parameters $N = 1$ and $M = 2$ to match the orders of the basis functions of the unknown system, i.e.,

$$\hat{y}(n) = a_1 x(n) + a_{12} x(n-1)x(n-2) + b_1 \hat{y}(n-1) + b_{11}\hat{y}^2(n-1) + b_{12}\hat{y}(n-1)\hat{y}(n-2) + c_{12}x(n-1)\hat{y}(n-2). \quad (23)$$

The SRV filter is then described by the following expression

$$\hat{y}(n) = a'_1 x(n) + a'_{12} x(n-1)x(n-2) + b'_1 \tanh[\hat{y}(n-1)] + b'_{11} \tanh[\hat{y}^2(n-1)] + b'_{12} \tanh[\hat{y}(n-1)] \tanh[\hat{y}(n-2)] + c'_{12} x(n-1) \tanh[\hat{y}(n-2)]. \quad (24)$$

An independent zero-mean Gaussian noise is added to the output signal, so that the output SNR is equal to 20dB. The step size is chosen equal to 0.025 and 100 independent runs are performed. With this set of specifics, the learning curves of RV and SRV filters are in practice coincident. Moreover, the mean-square errors, normalized to the power of the output signal and measured as mean values on the last 2000 time instants, result in -19.3 dB and -19.4 dB for the RV and SRV filters, respectively. These facts confirm that, for small input signals, the SRV filter gives a good approximation of the unknown system. Then, the RV and SRV filters are implemented using the coefficients in Table II, computed as the mean values on the 100 independent runs and the last 2000 time instants. The variance of the zero-mean Gaussian input signal is changed to 0.04 after $20\,000$ time instants, so that the input signal covers a range of about ± 0.80. Figure 1 illustrates the behavior of the two filters. Till to $20\,000$ time instants they give the same outputs, but after $20\,000$ time instants the RV filter becomes unstable while the SRV filter remains stable, with normalized mean square error (NMSE) equal to -18.1 dB at $40\,000$ time instants.

C. An identification experiment

We now consider an experiment involving real data generated by an electronic nonlinear feedback system (named "the silver box"), available in the literature for benchmarking in system identification [29]. The silver box describes a second order mechanical system with a nonlinear spring constant. In order to identify the system, one of the random odd multi-sine sequences composing the benchmarking data has been extracted. The input and output sequences are formed by 8700 samples (starting at sample $49\,270$). First, the input sequence has been normalized by its maximum value. Then, the input and output sequences have been periodically repeated ten times to form the signals used in the identification. The system has been identified with different recursive and non-recursive nonlinear filters, exploiting the output-error pseudo-NLMS in (21) and the standard NLMS algorithms, respectively. Table III provides the NMSE on the last 8700 samples for the following filters: recursive EMFN (REMFN), recursive Volterra (RV), stabilized recursive Volterra (SRV), recursive Legendre (RL), stabilized recursive Legendre (SRL), recursive Chebyshev (RC), stabilized recursive Chebyshev (SRC), bilinear filter (BIL), stabilized bilinear filter (SBIL), EMFN, Volterra (VOL), Legendre (LEG), and Chebyshev (CHEB). The recursive EMFN, Volterra, Legendre, and Chebyshev filters have order $P = 3$, parameters $N = 6$, $M = 6$ and 560 coefficients. The bilinear filter [15] has parameters $N = M = 23$ and 552 coefficients and is a particular case of recursive Volterra filter, as defined in this paper. The stabilized bilinear filter has been obtained as described in Section III, i.e., replacing $\hat{y}(n-i)$ with $\tanh[\hat{y}(n-i)]$ in the input-output relationship, as follows

$$\hat{y}(n) = \sum_{i=0}^{N} a_i x(n-i) + \sum_{j=1}^{M} b_j \tanh[\hat{y}(n-j)] + \sum_{i=0}^{N} \sum_{j=1}^{M} c_{i,j} x(n-i) \tanh[\hat{y}(n-j)]. \qquad (25)$$

The non-recursive filters have order $P = 3$, parameter $N = 12$ and 560 coefficients. For each filter, we have chosen the step-size that guarantees the best NMSE. Given the oscillatory nature of the output signal, the recursive filters are best fitted to model the nonlinear system. All recursive filters provide similar results, with the best results provided by the recursive Volterra and stabilized recursive Volterra filters. The stabilization does not influence the performance of the filters. On the contrary, in the non-recursive filters a memory length of 13 samples (i.e., $N = 12$) is insufficient for modeling the nonlinear system behavior. Performance similar to those of the recursive filters has been obtained with a Volterra filter with $N = 40$ and 13 244 coefficients.

V. Conclusions

In this paper, two novel subclasses of LIP nonlinear filters are described: RFLiP filters, that are universal approximators but potentially unstable, and SRFLiP filters, that give performance close to those of RFLiP filters and are always stable. While these filters already represent unknown systems with fewer coefficients than their finite memory counterparts, a study is underway for further reducing their complexity exploiting sparsity.

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