LOCALIZATION OF A MOBILE RIGID SENSOR NETWORK

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ABSTRACT

The relative positions of the sensors from one another in a rigid sensor network are known and locating the network reduces to obtaining its position, orientation angle, and translational and angular velocities with respect to a global coordinate frame from the measurements with anchors. Previous solution is computationally demanding that may not be suitable in a resource constrained environment. We propose a solution for this highly nonlinear estimation problem using the divide and conquer approach in the 2-D scenario. We first obtain from the measurements the sensor positions and velocities pretending no prior knowledge among them and then exploit their relative positions to estimate the unknown parameters. Methods are available for the first step. We focus on the second step and develop a closed-form solution through nuisance variables and nonlinear transformations. The proposed estimator is computationally attractive and has CRLB performance for Gaussian noise over the small error region.

Index Terms— Closed-form solution, localization, mobile rigid network, time and Doppler measurements

1. INTRODUCTION

Locating an object requires a network of sensors whose positions are known [1]. In practice the positions of the sensors are often not known when they are deployed. Self-localization of the sensor nodes is a promising technique to identify the positions of the sensors through message exchanges among the nodes and the anchors [2, 3]. Over the years many self-localization methods have been developed, mostly based on range measurements [4, 5].

Self-localization typically assumes the sensor node positions are independent of one another. In some situations, the relative positions of the sensors are known exactly [6–8]. A simple example is that the sensors are mounted on a fixed structure such as a vehicle. Consequently, locating the sensor nodes reduces to determining the position (translation vector) and the orientation (rotation matrix) of the entire network, and the translational and angular velocities as well if it is moving. By carefully exploiting the relative positions, the sensor locations can be identified with much better accuracy. The focus of this paper is to locate a rigid sensor network in which the relative positions of the nodes are known *a priori*.

A direct approach for the localization problem is to express the measurements in terms of the position vector, rotation matrix and velocities and formulate a cost function for optimization. This approach is not practical. First, the measurement equations are highly nonlinear with the unknowns coupled. Second, the elements in the rotation matrix are not free variables and must satisfy certain constraints [9].

In this paper, we propose an indirect approach to solve the problem in 2-D. The proposed approach obtains the sensor positions and velocities first, assuming there is no knowledge among them. Second, they are used to estimate the unknowns by exploiting the relative positions. Such an approach is called divide and conquer (DAC) in the literature [10]. Under Gaussian noise and independent measurements the DAC approach has been shown to achieve the CRLB accuracy, with the limitation that the thresholding effect may appear at a lower noise level.

Although the indirect approach simplifies the problem a little and the solution to the first step is available in the literature, the second step remains to be a nonlinear constrained optimization problem. We propose a two-stage processing in the second step through nuisance variables and nonlinear transformations to obtain a computationally attractive closed-form solution.

The proposed solution is different from the maximum likelihood estimator using iterative geometric descent [6] that is very computationally demanding and requires initial solution guesses. The method presented in [7] by building a dynamic system that evolves on the special Euclidean group is suitable only for near noise-free measurements. The orthogonal Procrustes problem technique proposed in [8] has suboptimum performance compared to the CRLB. All of them work for a stationary rigid network. For a moving rigid network, [11] requires the translational and angular velocities to be known. The solution from [12] uses sequential estimation and refinement technique. It could be computationally demanding and may not be suitable in a resource constrained environment.

The study here assumes range (TOA) and range rate (Doppler) observations and the proposed solution can be easily adopted to other forms of measurements.

We shall use the common notations that bold lowercase letter denotes column vector and bold uppercase letter repre-

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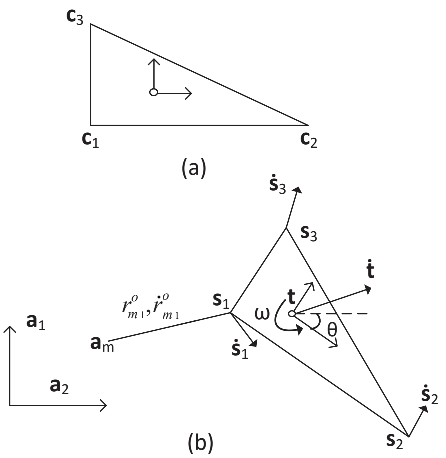

Fig. 1. Localization scenario.

sents matrix. $\mathbf{I}$ and $\mathbf{O}$ are identity and zero matrices, the size is 2 unless stated otherwise. $\otimes$ is the Kronecker product.

2. LOCALIZATION SCENARIO

As illustrated in Fig. 1, we are interested in determining the positions $\mathbf{s}_i \in \mathbb{R}^2$ and velocities $\dot{\mathbf{s}}_i \in \mathbb{R}^2$ of the sensor nodes, $i = 1, 2, \ldots, N$, using the range and range rate measurements

$$\begin{aligned} r_{mi} &= r^o_{mi} + v_{mi} \\ &= \|\mathbf{a}_m - \mathbf{s}_i\| + v_{mi} \end{aligned} \tag{1}$$

$$\dot{r}_{mi} = \frac{(\mathbf{s}_i - \mathbf{a}_m)^T}{r^o_{mi}} \dot{\mathbf{s}}_i + \dot{v}_{mi} \tag{2}$$

from a number of stationary anchors whose positions $\mathbf{a}_m \in \mathbb{R}^2$, $m = 1, 2, \ldots, M$, are known exactly. In (1) and (2), v_{mi} and $\dot{v}_{mi}$ are zero-mean Gaussian noise that can be correlated themselves and with each other. We assume the measurements from different sensors are independent. The collection of all the measurements forms the observation vector.

Unlike a typical sensor network localization problem, the relative positions of the sensors in a local reference, denoted by $\mathbf{c}_i$, are known exactly. Let the position of the local reference be $\mathbf{t} \in \mathbb{R}^2$ and the orientation angle be θ in the global coordinate frame. Then we have the relationship [8]

$$\mathbf{s}_i = \mathbf{Q}\mathbf{c}_i + \mathbf{t} \tag{3}$$

for the sensor positions. $\mathbf{Q}$ is the rotation matrix defined as

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{4}$$

The velocity relationship is [12]

$$\dot{\mathbf{s}}_i = [\omega]^\times \mathbf{Q}\mathbf{c}_i + \dot{\mathbf{t}} \tag{5}$$

where

$$[\omega]^\times = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \omega\,. \tag{6}$$

$\dot{\mathbf{t}}$ is the translational velocity and ω the angular velocity of the local reference for the rigid network.

Since the sensor positions and velocities must fulfill (3) and (5), the problem becomes the estimation of $\mathbf{t}$, θ, $\dot{\mathbf{t}}$ and ω. Note that through the parameterization in terms of θ, the constraints on the elements of $\mathbf{Q}$ is automatically satisfied. In this study, we only use the measurements between the sensors and the anchors. The range measurements between two sensors bear no additional information since the relative positions of the sensors are completely known.

3. NEW METHOD

Obtaining the unknown parameters directly from the measurements by putting (3) and (5) into (1) and (2) is difficult due to the highly nonlinear relationships and the coupling of the unknowns. We shall resort to the DAC approach by first obtaining $(\mathbf{s}_i, \dot{\mathbf{s}}_i)$ from the measurements and next estimating the unknowns using them. It has been shown in [10] that the DAC technique can yield the CRLB performance under Gaussian measurement noise, when the measurements for different sensors are uncorrelated.

The research of estimating $(\mathbf{s}_i, \dot{\mathbf{s}}_i)$ from the measurements is quite mature and many algorithms are available from literature, e.g. [12, 16]. We shall focus on the second step here.

3.1. Utilizing Initial Estimate

Let $(\hat{\mathbf{s}}_i, \hat{\dot{\mathbf{s}}}_i)$ be the solution from the first step with $(\mathbf{n}_{\mathbf{s}_i}, \mathbf{n}_{\dot{\mathbf{s}}_i})$ the estimation noise. From (3) we have [14]

$$\hat{\mathbf{s}}_i = \mathbf{Q}\mathbf{c}_i + \mathbf{t} + \mathbf{n}_{\mathbf{s}_i} = (\mathbf{c}_i^T \otimes \mathbf{I})\mathbf{\Gamma} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \mathbf{t} + \mathbf{n}_{\mathbf{s}_i} \tag{7}$$

where we have used the vectorization

$$\mathrm{vec}(\mathbf{Q}) = \mathbf{\Gamma} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tag{8}$$

and $\mathbf{\Gamma}$ is a 4×2 sparse matrix with the (1, 1), (2, 2) and (4, 1) elements equal to 1 and (3, 2) element -1. Similarly, from (5)

$$\hat{\dot{\mathbf{s}}}_i = [\omega]^\times \mathbf{Q}\mathbf{c}_i + \dot{\mathbf{t}} + \mathbf{n}_{\dot{\mathbf{s}}_i} = (\mathbf{c}_i^T \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})\mathbf{\Gamma} \begin{bmatrix} \omega\cos\theta \\ \omega\sin\theta \end{bmatrix} + \dot{\mathbf{t}} + \mathbf{n}_{\dot{\mathbf{s}}_i}. \tag{9}$$

In (7) and (9), $\mathbf{n}_{\mathbf{s}_i}$ and $\mathbf{n}_{\dot{\mathbf{s}}_i}$ are dependent on the range and range rate measurement noises.

Stacking (7) and (9) over i from 1 to N yields

$$\mathbf{d} = \mathbf{E}_1 \begin{bmatrix} \cos\theta \\ \sin\theta \\ \omega\cos\theta \\ \omega\sin\theta \end{bmatrix} + \mathbf{E}_2 \begin{bmatrix} \mathbf{t} \\ \dot{\mathbf{t}} \end{bmatrix} + \mathbf{n}\,. \tag{10}$$

$\mathbf{d}$ is the collections of $\hat{\mathbf{s}}_i$ and $\hat{\dot{\mathbf{s}}}_i$ over i,

$$\mathbf{E}_1 = \begin{bmatrix} (\mathbf{c}_1^T \otimes \mathbf{I})\mathbf{\Gamma} & \mathbf{O} \\ \mathbf{O} & \left(\mathbf{c}_1^T \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\mathbf{\Gamma} \\ \vdots & \vdots \\ (\mathbf{c}_N^T \otimes \mathbf{I})\mathbf{\Gamma} & \mathbf{O} \\ \mathbf{O} & \left(\mathbf{c}_N^T \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\mathbf{\Gamma} \end{bmatrix}, \mathbf{E}_2 = \mathbf{1} \otimes \mathbf{I}\,. \tag{11}$$

In $\mathbf{E}_2$ the sizes of $\mathbf{1}$ and $\mathbf{I}$ are N and 4 respectively. $\mathbf{n}$ is resulted from the estimation error of the first step and it can be approximated as a zero-mean Gaussian noise vector over the small error region since the measurement noise is Gaussian [13]. The covariance matrix is denoted by $\mathbf{R_n}$, which is block diagonal with individual block for each sensor since the measurements from different sensors are independent. It can be obtained based on the measurement noise covariance matrix and the estimates $\hat{\mathbf{s}}_i$ and $\hat{\dot{\mathbf{s}}}_i$ [12].

The unknowns θ and ω are embedded in $\mathbf{x} = [\cos\theta, \sin\theta, \omega\cos\theta, \omega\sin\theta]^T$. We propose to solve $\mathbf{x}$ instead by imposing two quadratic constraints

$$x_1^2 + x_2^2 = 1 \tag{12}$$

$$x_1 x_4 = x_2 x_3\,. \tag{13}$$

It can be shown that the weighted linear least squares [13] solution to (10) with the two quadratic constraints will yield the optimum accuracy as solving (θ, ω) directly from (10). This constrained optimization problem remains challenging to solve. We shall propose a computationally attractive closed-form solution to the problem.

3.2. Closed-form Solution

There are two sets of variables to be solved, $\mathbf{x}$ and $[\mathbf{t}^T, \dot{\mathbf{t}}^T]^T$, both appear linear in (10) with constraints imposed on the former only. In term of $\mathbf{x}$, the weighted least squares (WLS) solution for $[\mathbf{t}^T, \dot{\mathbf{t}}^T]^T$ with weighting $\mathbf{R_n}^{-1}$ is [13]

$$[\,\hat{\mathbf{t}}^T, \hat{\dot{\mathbf{t}}}^T\,]^T = \left(\mathbf{E}_2^T \mathbf{R_n}^{-1} \mathbf{E}_2\right)^{-1} \mathbf{E}_2^T \mathbf{R_n}^{-1} (\mathbf{d} - \mathbf{E}_1 \mathbf{x})\,. \tag{14}$$

Putting it into (10) yields a linear equation in $\mathbf{x}$ only,

$$\mathbf{h}_1 = \mathbf{G}_1 \mathbf{x} + \mathbf{n} \tag{15}$$

where $\mathbf{P} = \mathbf{I} - \mathbf{E}_2 \left(\mathbf{E}_2^T \mathbf{R_n}^{-1} \mathbf{E}_2\right)^{-1} \mathbf{E}_2^T \mathbf{R_n}^{-1}$, $\mathbf{h}_1 = \mathbf{P}\mathbf{d}$ and $\mathbf{G}_1 = \mathbf{P}\mathbf{E}_1$.

We shall propose a two-stage approach to solve (15) for $\mathbf{x}$ under the constraints (12) and (13). The first stage ignores the constraints to obtain $\mathbf{x}$. The second stage utilizes the constraints to construct another minimization process to improve the estimate. Once it is found, $[\mathbf{t}^T, \dot{\mathbf{t}}^T]^T$ is immediately available from (14). The proposed two-stage solution is new and different from those for non-rigid networks [15, 16].

3.2.1. Two-Stage Processing

1) Stage-1

We omit the constraints and the WLS solution is

$$\hat{\mathbf{x}} = (\mathbf{G}_1^T \mathbf{W}_1 \mathbf{G}_1)^{-1} \mathbf{G}_1^T \mathbf{W}_1 \mathbf{h}_1 \tag{16}$$

where $\mathbf{W}_1 = \mathbf{R_n}^{-1}$. Let us denote the estimation error as $\Delta\mathbf{x}$. Then $\mathrm{E}[\Delta\mathbf{x}\Delta\mathbf{x}^T] = \mathrm{cov}(\hat{\mathbf{x}}) = (\mathbf{G}_1^T \mathbf{W}_1 \mathbf{G}_1)^{-1}$.

2) Stage-2

We shall correct the stage-1 solution by taking the two constraints into account. It is more convenient to express these two constraints into different forms. Multiplying both sides of (13) by x_2 and substituting (12) for x_2^2 give

$$x_3 = (x_1 x_3 + x_2 x_4) x_1\,. \tag{17}$$

Multiplying both sides of (13) by x_1 and using (12) yield

$$x_4 = (x_1 x_3 + x_2 x_4) x_2\,. \tag{18}$$

We choose the parameter vector as $\boldsymbol{\beta} = [\omega\cos\theta, \omega\sin\theta]^T$ $=[x_3, x_4]^T$. The elements have direct mapping relationship with the unknowns θ and ω.

To utilize the constraint (17), we express the right side in terms of the elements of $\hat{\mathbf{x}}$ (i.e. $x_i = \hat{x}_i - \Delta x_i$) and obtain, after ignoring the second and third order errors,

$$\begin{aligned} x_3 = (\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4)\hat{x}_1 - (2\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4)\Delta x_1 \\ - \hat{x}_1\hat{x}_4\Delta x_2 - \hat{x}_1^2\Delta x_3 - \hat{x}_1\hat{x}_2\Delta x_4\,. \end{aligned} \tag{19}$$

Applying the same process gives the corresponding expression for (18).

We can now construct the matrix equation for stage-2 as

$$\mathbf{B}_2 \Delta\mathbf{x} = \mathbf{h}_2 - \mathbf{G}_2 \boldsymbol{\beta} \tag{20}$$

where

$$\mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 2\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4 & \hat{x}_1\hat{x}_4 & \hat{x}_1^2 & \hat{x}_1\hat{x}_2 \\ 0 & 0 & 0 & 1 \\ \hat{x}_2\hat{x}_3 & 2\hat{x}_2\hat{x}_4 + \hat{x}_1\hat{x}_3 & \hat{x}_1\hat{x}_2 & \hat{x}_2^2 \end{bmatrix},$$

$$\mathbf{h}_2 = \begin{bmatrix} \hat{x}_3 \\ (\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4)\hat{x}_1 \\ \hat{x}_4 \\ (\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4)\hat{x}_2 \end{bmatrix}, \ \mathbf{G}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{21}$$

The WLS solution for $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = (\mathbf{G}_2^T \mathbf{W}_2 \mathbf{G}_2)^{-1} \mathbf{G}_2^T \mathbf{W}_2 \mathbf{h}_2 \tag{22}$$

where the weighting matrix is from the covariance of $\hat{\mathbf{x}}$:

$$\begin{aligned} \mathbf{W}_2 &= [\mathbf{B}_2 (\mathbf{G}_1^T \mathbf{W}_1 \mathbf{G}_1)^{-1} \mathbf{B}_2^T]^{-1} \\ &= \mathbf{B}_2^{-T} (\mathbf{G}_1^T \mathbf{W}_1 \mathbf{G}_1) \mathbf{B}_2^{-1}\,. \end{aligned} \tag{23}$$

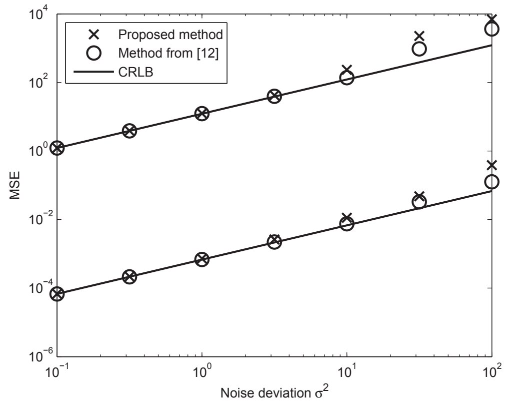

Fig. 2. Performance for orientation θ (deg, upper curves) and angular velocity ω (lower curves) estimations.

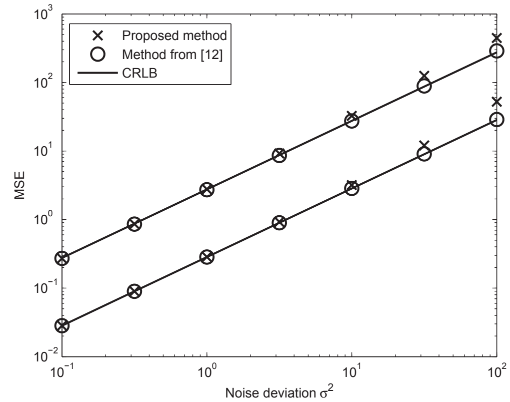

Fig. 3. Performance for position $\mathbf{t}$ (upper curves) and translational velocity $\dot{\mathbf{t}}$ (lower curves) estimations.

Finally, we can recover the estimates for θ and ω by

$$\hat{\omega} = \|\hat{\boldsymbol{\beta}}\|\mathrm{sgn}(\hat{\mathrm{x}}_1\hat{\mathrm{x}}_3 + \hat{\mathrm{x}}_2\hat{\mathrm{x}}_4) \tag{24}$$

$$\hat{\theta} = \mathrm{arctan2}(\hat{\beta}_2/\hat{\omega}, \hat{\beta}_1/\hat{\omega}) \tag{25}$$

where $\mathrm{arctan2}$ is the four-quadrant inverse tangent function. Updating $\mathbf{x}$ and putting it into (14) give the estimates for $\mathbf{t}$ and $\dot{\mathbf{t}}$. The final estimates of the sensor positions and velocities can now be obtained using (3) and (5).

Given that the initial estimates $(\hat{\mathbf{s}}_i, \hat{\dot{\mathbf{s}}}_i)$, $i = 1, 2, \cdots, N$ attain the CRLB when ignoring the prior knowledge of the relative sensor positions, we can show theoretically that the performance of the proposed closed-form solution in Section 3 approaches the CRLB accuracy over the small error region. This can be accomplished by expressing the covariance of $\hat{\omega}$ and $\hat{\theta}$ in terms of $\mathrm{cov}(\hat{\boldsymbol{\beta}})$ using (24) and (25). $\mathrm{cov}(\hat{\boldsymbol{\beta}})$ is equal to $(\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2)^{-1}$. After substituting (23) and noting that $\mathbf{W}_1 = \mathbf{R}_\mathbf{n}^{-1}$, the covariance of $\hat{\omega}$ and $\hat{\theta}$ equals the corresponding CRLB. Continuing in a similar manner we can also prove that $\hat{\mathbf{t}}$ and $\hat{\dot{\mathbf{t}}}$ also reach their CRLB accuracy.

4. SIMULATIONS

There are $M = 6$ anchors placed uniformly on the circle with $\mathbf{a}_m = 25[\cos\frac{2\pi}{M}(m-1), \sin\frac{2\pi}{M}(m-1)]^T$. Each sensor is able to acquire the measurements from all anchors. The sensor geometry is a square given by

$$\mathbf{C} = 5\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

where the columns are $\mathbf{c}_i$'s. It specifies the positions of the sensors with respect to the first. The rigid network has orientation $\theta = 20\,\mathrm{deg}$ and position $\mathbf{t} = [100\ 100]^T$ with respect to the global coordinate frame. The angular velocity is $\omega = 0.3$ rad/s and translational velocity is $\dot{\mathbf{t}} = [1, 1]^T$. For simplicity the covariance matrix of the distance measurements is set to be $\mathbf{R}_\mathbf{v} = \sigma^2\mathbf{I}$, the range rate measurements are uncorrelated with the distance measurements and have a covariance matrix $\mathbf{R}_{\dot{\mathbf{v}}} = 0.1\mathbf{R}_\mathbf{v}$. The number of ensemble runs is $L = 2000$.

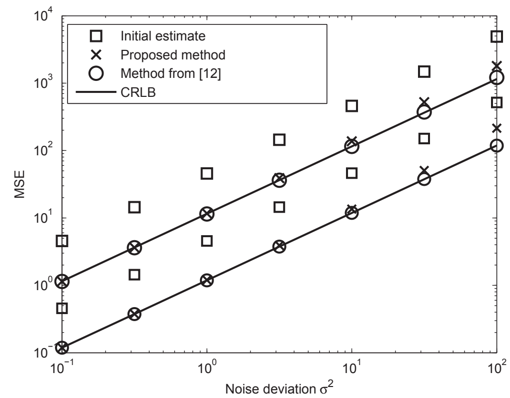

Fig. 4. Performance for sensor positions (upper curves) and velocities (lower curves) estimations.

The first step uses the algorithm based on [12,16] to obtain the initial sensor position and velocity.

4.1. Accuracy Comparison

Fig. 2 shows the results of the orientation angle (upper curves) and angular velocity (lower curves) using the proposed method and the method from [12]. The proposed method reaches the CRLB [12] performance in the small error region and has comparable accuracy with the method in [12]. It deviates from the bound a little earlier as the noise level increases. We have similar observations for the position and translational velocity estimates as shown in Fig. 3.

Fig. 4 illustrates the performance in terms of the sensor position and velocity estimates. When we exploit the known relative sensor locations, the accuracy is much better than without (initial estimate) in both positions and velocities. We

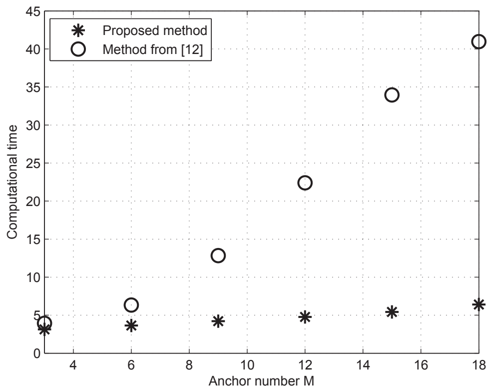

Fig. 5. Computational times vs anchor number M.

generated the CRLB for the sensor positions and velocities using the CRLB for the four parameters through the relationships (3) and (5) [13]. The proposed algorithm achieves the optimum performance when the noise level is not significant and it performs comparably with the algorithm from [12] unless the noise level is high.

4.2. Computation Time Comparison

The proposed method has the benefit of lower computational complexity than the one in [12]. Fig. 5 illustrates the computation times (millisecond) of each ensemble run obtained from MATLAB implementation of two methods, when the number of anchors M varies from 3 to 18. The computational advantage of the proposed method is obvious when the number of anchors is large. The proposed method is a good alternative when the noise power is not significant.

5. CONCLUSION

A new estimator for locating a moving rigid sensor network in 2-D is presented. The proposed estimator uses the DAC approach where initial sensor positions and velocities are estimated from the measurements first and the unknowns are deduced from them by utilizing the known relative positions of the sensors. We have developed a computationally attractive closed-form solution that involves two quadratic constraints. Simulations show good performance of the proposed estimator and it achieves the CRLB accuracy for Gaussian noise over the small error region. It requires less computation than the previous solution from [12], which is particularly important when operating under a resource constrained environment. The work presented here focuses on the 2-D scenario. Extension to the 3-D case is under investigation.

6. REFERENCES

[1] N. Patwari, J. N. Ash, S. Kyperountas, A. O. Hero, R. L. Moses, and N. S. Correal, "Locating the nodes: cooperative localization in wireless sensor networks," *IEEE Signal Process. Mag.*, vol. 22, no. 4, pp. 54–69, Jul. 2005.

[2] J. Teng, R. Zhou, and Y. Zhang, "Cooperative simultaneous localization and tracking for mobile ad-hoc sensor networks," in *Proc. EUSIPCO 2015*, Nice, France, Aug. 2015, pp. 2197–2201.

[3] R. L. Moses, D. Krishnamurthy, and R. M. Patterson, "A self-localization method for wireless sensor networks," *EURASIP J. Applied Signal Process.*, no. 4, pp. 348–358, 2003.

[4] A. T. Ihler, J. W. Fisher, R. L. Moses, and A. S. Willsky," Nonparametric belief propagation for self-localization of sensor networks," *IEEE J. Sel. Areas Commun.*, vol. 23, no. 4, pp. 809–819, Apr. 2005.

[5] P. Biswas, T.-C. Liang, K.-C. Toh, Y. Ye, and T.-C. Wang, "Semidefinite programming approaches for sensor network localization with noisy distance measurements," *IEEE Trans. Autom. Sci. Eng.*, vol. 3, no. 4, pp. 360–371, Oct. 2006.

[6] A. Alcocer, P. Oliveira, A. Pascoal, and J. Xavier, "Estimation of attitude and position from range-only measurements using geometric descent optimization on the special Euclidean group," in *Proc. 9th Int. Conf. Inf. Fusion*, Jul. 2006, pp. 1–8.

[7] A. Alcocer, P. Oliveira, A. Pascoal, R. Cunha, and C. Silvestre, "A dynamic estimator on *SE*(3) using range-only measurements," in *Proc. 47th IEEE Conf. Decision & Control*, Dec. 2008, pp. 2302–2307.

[8] S. P. Chepuri, G. Leus, and A.-J. van der Veen, "Rigid body localization using sensor networks," *IEEE Trans. Signal Process.*, vol. 62, no. 18, pp. 4911–4924, Sep. 2014.

[9] J. Diebel, "Representing attitude: Euler angles, unit quaternions, and rotation vectors," *Matrix*, vol. 58, pp. 15–16, 2006.

[10] J. S. Abel, "A divide and conquer approach to least-squares estimation," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 26, no. 2, pp. 423–427, Mar. 1990.

[11] S. P. Chepuri, A. Simonetto, G. Leus and A. J. Van der Veen, "Tracking position and orientation of a mobile rigid body," in *IEEE 5th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, Dec. 2013, pp. 37–40.

[12] S. Chen and K. C. Ho, "Accurate localization of a rigid body using multiple sensors and landmarks," *IEEE Trans. Signal Process.*, vol. 63, no. 24, pp. 6459-6472, Dec. 2015.

[13] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. Englewood Cliffs, NJ: Prentice Hall, 1993.

[14] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Presss, Cambridge, 1991.

[15] Y. T. Chan and K. C. Ho, "A simple and efficient estimator for hyperbolic location," *IEEE Trans. Signal Process.*, vol. 42, no. 8, pp. 1905–1915, Aug. 1994.

[16] K. C. Ho and W. Xu, "An accurate algebraic solution for moving source location using TDOA and FDOA measurements," *IEEE Trans. Signal Process.*, vol. 52, no. 9, pp. 2453–2463, Sep. 2004.