On the Existence of the Band-Limited Interpolation of Non-Band-Limited Signals

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***Abstract*—The distribution theory serves as an important theoretical foundation for some approaches arose from the engineering intuition. Particular examples are approaches based on the delta-"function". In this work, we show that the usual construction of a band-limited interpolation (BLI) of signals "vanishing" at infinity (e.g., in [1], [2]), using the delta-"function", is erroneous, both in the distributional sense and in the tempered distributional sense. The latter sense is in particular important for analyzing the frequency behaviour of that method – the aliasing error and the truncation error. Furthermore, we show that it is possible to construct a BLI without using the delta-"function". This can in particular be done easily for the space of signals having integrable frequencies. If one consider another notion of band-limited functions, a BLI can even be given for the space of continuous signals "vanishing" at infinity. For the space of continuous signals, we answer the question whether there exists a BLI negatively.**

***Index Terms*—Band-limited interpolation, Band-limited signals, (Tempered) Distributions, Sampling, Divergence**

I. Introduction

Signal processing is nowadays performed mostly with digital processors, while the physical quantities of the real world remain analog. Therefore, the conversion of continuous-time signals into discrete-time signals, i.e., *sampling*, and the conversion of discrete-time signals into continuous-time signals, i.e., *interpolation* (in case that the interpolation yields the original signal, we speak of *reconstruction*), are essential. The *Shannon's sampling series (SSS)* [3] is probably one of the prominent example of a reconstruction (i.e., sampling and interpolation) method. Its significance in the signal processing is founded by the fact, that it ensures a perfect reconstruction of a band-limited square-integrable signal from its samples taken at a sufficiently high rate – the Nyquist rate. Since this initial result, many sampling theorems of different directions have been developed, aiming to broaden the signal classes, and to examine the convergence/divergence of the SSS for several signal classes more closely. Some excellent overviews can be found in e.g., [4], [5], [6], [7], [8], [9].

In standard literature, the use of the so-called delta-"function" or Dirac-comb to describe the process of sampling and reconstruction/interpolation by means of the SSS has become common practice. The following interpolation method constitutes a prototypical example: Given a signal f – an arbitrary function. Some authors claim (for instance: in pp. 52 in [2], in pp. 114 in [1]), that one can construct the so-called *band-limited interpolation (BLI)* f_π of f, i.e., the band-limited signal $f_\pi \in \mathcal{PW}_\pi^1$ (cf. Section II), for which $f(k) = f_\pi(k)$, $\forall k \in \mathbb{Z}$, holds. Notice that it is sufficient, only to consider the band-limit π, since usually any other band-limit yields from this case by simple rescaling. To construct the BLI, they proposed the following steps: First, multiply the signal f with the Dirac comb $\sum_{k=-\infty}^{+\infty} \delta(t-k)$ yielding $f_{\uparrow\uparrow}(t) = f(t)\sum_{k=-\infty}^{+\infty} \delta(t-k)$. Subsequently, convolve $f_{\uparrow\uparrow}$ with the ideal low-pass $h(t) = \sin(\pi t)/\pi t$, resulting the signal:

$$f_\pi(t) := \sum_{k=-\infty}^{\infty} f(k) \tfrac{\sin(\pi(t-k))}{\pi(t-k)}. \qquad (1)$$

Besides, those authors claims that the FT of f and f_π can not in general be identical, and accordingly also f and f_π, since aliasing might occur in the process of constructing f_π. In particular, this claim is allowed in turn by another claim, that the FT of $f_{\uparrow\uparrow}$ can be seen as the periodization of f by period π. There are some issues needs to be concerned with all of the previous claims. Firstly, the use of the delta-"function", which constitutes the building block of the Dirac-comb, has clearly to be founded rigorously, in the sense that the distribution theory (cf. Section II) has to be involved therein. Secondly, it remains unclear in which form the limit (1) does exist. Although in some of those literature (e.g., [2] and [1]), the basic of distribution theory was introduced, there was no effort made to ascertain, whether those steps to construct a BLI mentioned previously is correct in the distributional sense. Thirdly, the FT can only be defined for the restrictive class of distributions, viz., the class of tempered distributions (cf. Section II). Hence, it should also be investigate, whether the discussed BLI is also correct, not only in the distributional sense, but also in the tempered distributional sense, since a notion of FT can be given in the latter sense. We shall give a discussion on this aspect in Section IV.

In this work, we aim to examine those issues by means of techniques inspired by the *Banach-Steinhaus Theorem (BST)* [10] (cf. Theorem 1). The BST constitutes one of the cornerstones of functional analysis. It gives a powerful tool to proof results regarding to the divergence of certain (approximation)

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processes. For instance, it was shown in [11] by means of that technique, that there exists a signal $f \in \mathcal{PW}_\pi^1$, such that the peak value of its Shannon sampling series diverges. There, it was also shown that this behaviour holds for a large class of reconstruction processes. Furthermore, the BST implied that the set of such signals, for which the previously mentioned results holds, is large in the topological sense. As similar, the BST was used in [12] to show some tightenings of the Riemann-Lebesgue Lemma, i.a. the fact that band-limited signals have typically arbitrarily slow decay, and that the FT of integrable signals have typically worst continuity/smoothness behaviour. By means of that technique, we will specifically show that the BLI given in [1], [2] is actually erroneous, if interpreted both, in distributional (cf. Theorems 2 and 3) and tempered distributional sense (cf. Theorem 4). We will also propose in this work a way to construct a BLI of signals in the relatively small space (if compared to $\mathcal{C}_0$, the space of signals "vanishing" at infinity, and $\mathcal{C}$, the space of continuous signals, cf. (10)) $\mathcal{PW}^1$ of signals having integrable frequencies (cf. (6) and (7)). In case that one considers another notion of band-limitedness (cf. (10)), the BLI for $\mathcal{C}_0$ can also be constructed (cf. (13)). Motivated by that, we aim also to give an answer, whether a BLI in any form for the large class of signals $\mathcal{C}$ may exist. In particular, we will abnegate the existence of BLI in any form for $\mathcal{C}$. At last, we mention a question which we aim to answer in the subsequent work: In which case does the distribution theory gives a more general convergence statement for BLI than that of the classical theory of analytic functions.

II. Notations

An *operator* denotes a linear mapping between vector spaces, and a *functional* an operator between vector space and $\mathbb{C}$. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be normed spaces, and $T : \mathcal{X}_1 \to \mathcal{X}_2$. The *norm of the operator* T is given by:

$$\|T\| := \sup_{\|x\|_{\mathcal{X}_1} \neq 0} \frac{\|Tx\|_{\mathcal{X}_2}}{\|x\|_{\mathcal{X}_1}} = \sup_{\|x\|_{\mathcal{X}_1} \leqslant 1} \|Tx\|_{\mathcal{X}_2} = \sup_{\|x\|_{\mathcal{X}_1} = 1} \|Tx\|_{\mathcal{X}_2}$$

An operator is said to be bounded, if its norm is finite.

We denote the *space of continuous functions* on $\mathbb{R}$ by $\mathcal{C}$, and the *space of continuous functions f vanishing at "infinity"*, i.e., $\lim_{|t|\to\infty} |f(t)| = 0$, by $\mathcal{C}_0$. Equipped by the *supremum/maximum norm*: $\|\cdot\|_\infty : \mathcal{C}_0 \to \mathbb{R}_0^+,\ f \mapsto \sup_{t\in\mathbb{R}} |f(t)|$, it can be shown that $\mathcal{C}_0$ is a Banach space. In this paper, we also work with the following subspaces of $\mathcal{C}$ containing of *continuous functions with moderate slow growth*: $\mathcal{C}_\alpha := \{f \in \mathcal{C} :\ \sup_{t\in\mathbb{R}} |f(t)|\, e^{-\alpha|t|} < \infty\}$, where $\alpha > 0$. Equipped with the norm $\|f\|_{\mathcal{C}_\alpha} := \sup_{t\in\mathbb{R}} |f(t)|\, e^{-\alpha|t|}$, one can show that $\mathcal{C}_\alpha$ is a Banach space.

The space of test functions on the real line, which denotes simply the space of smooth functions, i.e., infinitely often differentiable functions on the real line, with compact support, viz. the set of points where the function is not zero-valued is bounded and closed, denoted by $\mathcal{D}$, and the *space of rapidly decreasing smooth functions*, or *Schwartz space* by $\mathcal{S}$, defined formally by $\mathcal{S} := \{f \in \mathcal{C}^\infty :\ \sup_{t\in\mathbb{R}} |t^\alpha f^{(\beta)}(t)| < \infty\}$, where $\mathcal{C}^\infty$ denotes the *space of infinitely differentiable functions* on the real line. By $\mathcal{D}'$ we denote the space of all continuous functionals on $\mathcal{D}$. Its elements are called *distribution*. Given a function $g : \mathbb{R} \to \mathbb{C}$, we say that g can be *interpreted as a distribution*, if the functional $\phi \mapsto \int_{-\infty}^{+\infty} \phi(t)g(t)\mathrm{d}t$ defines a distribution. In particular, the class of such functions coincides with the *class of locally integrable functions*, i.e., functions which are each integrable on all compact (i.e. bounded and closed) subsets of $\mathbb{R}$, i.e., $L^1_{\text{loc}} := \{f \text{ measurable} \mid \forall K \subset \mathbb{R} \text{ compact} :\ \int_K |f(t)|\,\mathrm{d}t < \infty\}$. As usual, the notion of *convergence in $\mathcal{D}'$* is given as follows: a sequence $\{\mathrm{T}_n\}$ in $\mathcal{D}'$ is said to converge to $\mathrm{T} \in \mathcal{D}'(\mathbb{R})$, if $\lim_{n\to\infty} \mathrm{T}_n(\phi) = \mathrm{T}(\phi)$, for all $\phi \in \mathcal{D}$. Let be $\mathrm{T} \in \mathcal{D}'$, we write the action of a distribution on a test function ϕ as usual by $\langle \mathrm{T}, \phi\rangle := \mathrm{T}(\phi)$. Under the term *tempered distribution*, we understand a functional on $\mathcal{S}$. The *space of all tempered distributions* is denoted by $\mathcal{S}'$. We denote the action of a tempered distribution $\mathrm{T} \in \mathcal{S}'$ on a Schwartz function $\varphi \in \mathcal{S}$ also by $\langle \mathrm{T}, \varphi\rangle$. The convergence of a sequence in $\mathcal{S}'$ is analogue to the convergence of sequence in $\mathcal{D}'$. The well-known advantage of tempered distributions, is that one can define the Fourier transform (FT) $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ of a tempered distribution by means of the equation $\langle \mathcal{F}\mathrm{T}, \phi\rangle =: \langle \mathrm{T}, \mathcal{F}\phi\rangle$, where ϕ goes over all $\mathcal{S}$. Under *delta-distribution* $\delta(t-x)$ or $\delta(\cdot - x)$, where $x \in \mathbb{R}$, we understand the continuous functional $\mathcal{D} \ni \varphi \mapsto \langle \delta(\cdot - x), \varphi\rangle = \varphi(x)$. Furthermore, for $x \in \mathbb{R}$, the delta-distribution can also be seen as an element of $\mathcal{S}'$. For more detailed treatment of distribution theory, we refer to e.g., [13], [14].

In standard literature, the notion of *band-limitedness* is vaguely given. We find more convenient to formalize that notion as follows: A signal f is said to be band-limited to a band-limit $\omega_g > 0$, if there exists $\hat{f} \in L^1([-\omega_g, \omega_g])$, for which $f(t) = (1/2\pi)\int_{-\omega_g}^{\omega_g} \hat{f}(\omega)e^{i\omega t}\mathrm{d}\omega$, i.e., f is the inverse Fourier transform (IFT) of $\hat{f}$. The space of such band-limited functions is denoted by $\mathcal{PW}^1_{\omega_g}$. We remark that there is another notion of band-limitedness (cf. (10)) Throughout this work, we mostly consider signals with band-limit π, since all the results can be transferred to any other band-limit by some simple rescaling. In this work, we shall also concern with the space $\mathcal{PW}^1 := \{f \in \mathcal{C}_0 :\ f = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega t}\mathrm{d}\omega,\ \hat{f} \in L^1(\mathbb{R})\}$. Further, $\mathcal{PW}^1$ is equipped with the norm $\|f\|_{\mathcal{PW}^1} := \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{f}(\omega)|\mathrm{d}\omega$, which makes this space a Banach space.

Let $\mathcal{B}$ be a Banach space, a set $\mathcal{M} \subseteq \mathcal{B}$ is said to be *nowhere dense* if the inner of the closure of $\mathcal{M}$ is empty. A set $\mathcal{M} \subseteq \mathcal{B}$ is said to be of 1st category, if it can be represented as a countable union of nowhere dense sets. The complement of a set of 1st category is defined as a *residual set*. Topologically, sets of 1st category can be seen as a small set. Accordingly, residual sets, each as a complementary set of a set of 1st category, can be seen as a large set. The Baire category Theorem ensures that this categorization of sets of a Banach spaces is non-trivial, by showing that the whole Banach space $\mathcal{B}$ is not small in this sense, or can even not be "approximated" by such sets, i.e., it can not be written as a countable union of sets of 1st category, and that the residual sets are dense in $\mathcal{B}$, and closed under countable intersection. A property that holds for a residual subset of $\mathcal{B}$ is called a *generic property*. A generic property might not holds for all elements of $\mathcal{B}$, but for "typical" elements of $\mathcal{B}$. The so-called Banach-Steinhaus Theorem [10], which is one of the central results in functional analysis, constitutes a consequence of the Baire category Theorem. One of its version can be expressed as follows:

Theorem 1 (The Principle of Condensation of Singularity): *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces. Given a family Φ of bounded operators between $\mathcal{B}_1$ and $\mathcal{B}_2$. If it holds $\sup_{\mathrm{T}\in\Phi}\|\mathrm{T}\|_{\mathcal{B}_1\to\mathcal{B}_2} = \infty$, then there exists an $x_* \in \mathcal{B}_1$ for which $\sup_{\mathrm{T}\in\Phi}\|\mathrm{T}x_*\|_{\mathcal{B}_1} = \infty$. Furthermore, the set of such x_* is a residual set in $\mathcal{B}_1$.*

The principle of condensation of singularity gives a powerful tool for proving divergence results. The corresponding procedure can be formulated as follows: Firstly, rewrite the problem s.t. Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ (mostly $\mathcal{B}_2 = \mathbb{C}, \mathbb{R}$) and a family of bounded operators Φ between $\mathcal{B}_1$ and $\mathcal{B}_2$ occurs in its reformulation. Secondly, show that $\sup_{\mathrm{T}\in\Phi}\|\mathrm{T}\|_{\mathcal{B}_1\to\mathcal{B}_2} = \infty$, which is usually easier than observing the behaviour of $\|\mathrm{T}x\|_{\mathcal{B}_2}$, for every $\mathrm{T} \in \Phi$, and $x \in \mathcal{B}_1$. Finally, one can immediately infer that there exists not only an $x \in \mathcal{B}_1$ s.t. the divergence result holds, rather it holds for typical elements of $\mathcal{B}_1$. For more detailed treatment of the Baire category Theorem, and the Banach-Steinhaus Theorem, we refer to [15], [16], [17]

III. On the Existence of a BLI of $\mathcal{C}_0$ in $\mathcal{D}'$

First, we continue the discussion given in the 2nd paragraph in the introduction. Instead of considering all possible signals, we restrict ourselves to the class $\mathcal{C}_0$. We find the following heuristic steps for constructing a BLI more rigorously than that, mentioned in the introduction:

1) Although the multiplication of distributions is generally not defined, we give the multiplication of $f \in \mathcal{C}_0$ with the Dirac comb[1] $\sum_{k=-\infty}^{+\infty}\delta(t-k)$ as the distribution:
$$f_{\uparrow\uparrow}(t) = \sum_{k=-\infty}^{+\infty} f(k)\delta(t-k), \qquad (2)$$
where above expression has to be understood as the limit of the sequence of distributions $\{f_{\uparrow^N}\}$, where each of the members is given by $f_{\uparrow^N}(t) = \sum_{k=-N}^{N} f(k)\delta(t-k)$.
2) For each $N \in \mathbb{N}$, the convolution of $f_{\uparrow^N}$ with the ideal low-pass signal $h(t) = \sin(\pi t)/\pi t$ can heuristically be given as the distribution:
$$S_N f := \sum_{k=-N}^{N} f(k)\tfrac{\sin(\pi(t-k))}{\pi(t-k)}. \qquad (3)$$

Furthermore, it is not hard to see that f, h, and $S_N f$, $\forall N \in \mathbb{N}$, can be interpreted as distributions, since they are locally integrable. So, it stands clear to interpret (1) in the distributional sense as the limit of (3). The following Theorem asserts, that this anyway does not make sense:

Theorem 2: *There exists a function $f \in \mathcal{C}_0$, such that $f_{\uparrow(N)} * h_\pi$ does not converge in the distributional sense. Specifically, there exists $\phi_* \in \mathcal{D}$, s.t. $\lim_{N\to\infty}|\langle S_N f_*, \phi_*\rangle| = \infty$.*

Proof: First, take an arbitrary function $g_* \in \mathcal{D}$, with $\operatorname{supp}(g_*) \subseteq [-1/2, 1/2]$, $g_* \geqslant 0$, and $g(0) = 1$. For instance the function g_* with $g_*(t) := \exp\left(1 + \frac{-(1/4)}{(1/4)-t^2}\right)$, for $|t| < 1/2$, and $g_*(t) = 0$ else, fulfills those requirements. By means of g_*, define next the function $f_*(t) := \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{\log(1+l)}g_*(t-l)$. Obviously, f_* is continuous and "vanishes" at infinity.

Involving the properties of g_*, the samples of f_* on time instances $k \in \mathbb{Z}$ yields:
$$f_*(k) = \tfrac{(-1)^{k+1}}{\log(1+k)}, \text{ for } k \in \mathbb{N},\ f_*(k) = 0, \text{ for } k \in \mathbb{Z}\backslash\mathbb{N}. \quad (4)$$
Now, for each $N \in \mathbb{N}$, and every $t \in \mathbb{R}\backslash\mathbb{Z}$, we may compute:
$$\sum_{k=-N}^{N} f_*(k)\tfrac{\sin(\pi(t-k))}{\pi(t-k)} = \sum_{k=1}^{N}\tfrac{(-1)^{k+1}}{\log(1+k)}\tfrac{\sin(\pi(t-k))}{\pi(t-k)} = \tfrac{\sin(\pi t)}{\pi}\sum_{k=1}^{N}\tfrac{(-1)^k(-1)^{k+1}}{\log(1+k)}\tfrac{1}{(t-k)},$$
where the 1st equality follows by (4), and the 2nd equality follows from the application of the addition Theorem, and from the identity $\cos(\pi k) = (-1)^k$, $k \in \mathbb{N}$. Thus, for $t \in (0,1)$, we obtain the following simple estimate:
$$\sum_{k=-N}^{N} f_*(k)\tfrac{\sin(\pi(t-k))}{\pi(t-k)} = \tfrac{\sin(\pi t)}{\pi}\sum_{k=1}^{N}\tfrac{1}{\log(1+k)}\tfrac{1}{(-t+k)} > \tfrac{\sin(\pi t)}{\pi}\sum_{k=1}^{M}\tfrac{1}{\log(1+k)}\tfrac{1}{(1+k)}.$$
Since for each $k \in \mathbb{N}$, $1/[(1+x)(\log(1+x))]$ is strictly monotonically decreasing on $[k, k+1]$, and hence the area under its curve is strictly smaller than the area of the rectangular on the interval $[k, k+1]$ with height $1/[(1+k)(\log(1+k))]$, we may continue above estimation as follows:
$$\sum_{k=-N}^{N} f_*(k)\tfrac{\sin(\pi(t-k))}{\pi(t-k)} > \tfrac{\sin(\pi t)}{\pi}\sum_{k=1}^{N}\int_k^{k+1}\tfrac{\mathrm{d}x}{(1+x)\log(1+x)} = \tfrac{\sin(\pi t)}{\pi}\log\left(\tfrac{\log(2+N)}{\log 2}\right),$$
where the equality follows by combining the integrals and by substituting $u := \log(1+x)$.

Now take a test function $\phi_* \in \mathcal{D}$, with $\phi_* \geqslant 0$, $\operatorname{supp}\phi_* \subset [0,1]$, and $\phi_* = 1$, for every $t \in [1/4, 3/4]$. For instance, one may take a suitable *mesa function*. For each $N \in \mathbb{N}$, we may compute the action $\langle S_N f_*, \phi_*\rangle$ of $S_N f_*$ to ϕ_* by:
$$\int_0^1 (S_N f_*)(t)\phi_*(t)\mathrm{d}t > \tfrac{1}{\pi}\log\left(\tfrac{\log(2+N)}{\log 2}\right)\int_0^1 \sin(\pi t)\phi_*(t)\mathrm{d}t = \tfrac{2}{\pi}\log\left(\tfrac{\log(2+N)}{\log 2}\right)\cos\left(\tfrac{\pi}{4}\right). \quad (5)$$
■

[1] The Dirac comb has basically to be seen as the limit of the sequence of the distributions $\left\{\sum_{k=-N}^{N}\delta(t-k)\right\}$, and hence also itself a distribution, where δ denotes the delta-distribution

Thus $\lim_{N\to\infty} |\langle S_N f_*, \phi_*\rangle| = \infty$.

Reading above Theorem, one may think optimistically, that the case where the expression (1) is senseless for a function $f \in \mathcal{C}_0$ might occur seldom. As the following Theorem asserts, this is actually not the case:

Theorem 3: *The set of functions $f \in \mathcal{C}_0$, for which $\{f_{\uparrow(N)} * h_\pi\}$ does not converges in the distributional sense is a residual set in $\mathcal{C}_0$*

Proof: Define for each $N \in \mathbb{N}$ the functional $\Psi^{(N)}_{\phi_*} : \mathcal{C}_0 \to \mathbb{C}, f \mapsto \langle S_N f, \phi_*\rangle$. For each $N \in \mathbb{N}$, it is not hard to see that $\|\Psi^{(N)}_{\phi*}\|$ is finite. Furthermore, Theorem 2 asserts that $\{\|\Psi^{(N)}_{\phi_*}\|\}$ is unbounded. Specifically, by (5), we have for each $N \in \mathbb{N}$, $\left\|\Psi^{(N)}_{\phi_*}\right\| \geqslant \frac{2}{\pi}\log\left(\frac{\log(2+N)}{\log 2}\right)\left|\cos\left(\frac{\pi}{4}\right)\right|$, and accordingly $\sup_{N\in\mathbb{N}}\|\Psi^{(N)}_{\phi_*}\| = \infty$. Thus, we can conclude by Corollary 1 that: $\limsup_{N\to\infty}\|\Psi^{(N)}_{\phi_*} f\| = \infty$, and $\limsup_{N\to\infty}|\langle S_N f, \phi_*\rangle| = \infty$, for f from a residual set in $\mathcal{C}_0$, as desired. ∎

Remark 1: For signals in $\mathcal{PW}^1$, which is smaller than $\mathcal{C}_0$, it is not hard to construct a BLI. Further, it is unnecessary to involve the distribution theory: For $f \in \mathcal{PW}^1$, define the function:

$$f_\pi := \widecheck{F_\pi},\ F_\pi(\omega) := \begin{cases} \sum_{k=-\infty}^{\infty} \hat{f}(\omega + k2\pi), & |\omega| \leqslant \pi \\ 0, & \text{else.} \end{cases} \tag{6}$$

It can be shown, the infinite series determining F_π converges almost everywhere, and that $F_\pi \in L^1(\mathbb{R})$. Observe that $F_\pi \in \mathcal{PW}^1_\pi$. Further, by computations, involving Fubini-Tonelli's Theorem, one can show that the samples of $\widecheck{F_\pi}$ and f coincides, if taken at the points $\mathbb{Z}$, i.e., $\widecheck{F_\pi}(k) = f(k)$, $\forall k \in \mathbb{Z}$, which gives the last hints, that f_π is a BLI of $f \in \mathcal{PW}^1$. A more concrete representation of f_π can locally be derived: Since $f_\pi \in \mathcal{PW}^1_\pi$, the values of f_π and f coincides at the corresponding sampling points, and the SSS for $\mathcal{PW}^1_\pi$ converges locally uniformly [18], it holds for every $T > 0$:

$$\lim_{N\to\infty}\max_{t\in[-T,T]}\left|f_\pi(t) - \sum_{k=-N}^{N} f(k)\tfrac{\sin(\pi(t-k))}{\pi(t-k)}\right| = 0. \tag{7}$$

However, if one substitute the interval $[-T, T]$ by $\mathbb{R}$, above series might diverges [11].

IV. On the Existence of BLI of $\mathcal{C}_\alpha$ in $\mathcal{S}'$

To be able to investigate the statement concerning to the aliasing effect, occurring in the construction of BLI mentioned in the introduction, it is desired to analyze the spectral properties of the expressions (2) and (3). However, as is well-known, FT can not be defined generally for distributions. Rather, to be able to do this operation, one has to restrict the class of those generalized functions, to the class of tempered distributions. Motivated by this discussions, we aim in this section to answer the questions, whether the expressions (2) and (3) can be interpreted as tempered distributions, where f is contained in a subspace of $\mathcal{C}$.

Firstly, we want to answer the question, whether (2) converges in the tempered distributional sense. Specifically, we want to see, whether the sequence $\langle f_{\uparrow^N}, \varphi\rangle = \sum_{k=-N}^{N} f(k)\langle\delta(\cdot - k), \varphi\rangle = \sum_{k=-N}^{N} f(k)\varphi(k)$, where f is contained in the signal space of interests, converges, for every $\varphi \in \mathcal{S}$. It is not hard to see that (2) converges in that sense, for $f \in \mathcal{C}_0$. Now, we aim to see whether the expression (2) makes sense for the following larger subspace $\mathcal{C}_\alpha$ of $\mathcal{C}$ containing functions with moderate slow growth, where $\alpha > 0$. The corresponding answer is given in the following Theorem:

Theorem 4: *Let be $\varphi \in \mathcal{S}$, $\varphi \not\equiv 0$, with supp($\hat{\varphi}$) $\subset [-\omega_g, \omega_g]$, where $0 < \omega_g < \pi$. Then, the set of all $f \in \mathcal{C}_\alpha$, for which:*

$$\limsup_{N\to\infty}\left|\sum_{k=-N}^{N} f(k)\varphi(k)\right| = \infty \tag{8}$$

holds, is a residual set in $\mathcal{C}_\alpha$.

Proof: For an arbitrary $\varphi \in \mathcal{S}$, and $N \in \mathbb{N}$, it is not hard to see that the functional $\mathrm{T}_{N,\varphi}$ on $\mathcal{C}_\alpha$, which is given by $f \mapsto \sum_{k=-N}^{N} f(k)\varphi(k)$, is continuous. Choose a function $g \in \mathcal{D}$, $g \neq 0$, with $\operatorname{supp} g \subset [-1/2, 1/2]$, and $g(0) = 1$. Further take an $\varphi \in \mathcal{S}$ fulfilling the conditions given in the Theorem. Subsequently, define by those choices the function f_φ on $\mathbb{R}$ by $f_\varphi(t) = \sum_{k=-\infty}^{+\infty}\exp(\alpha|k|)\exp(-i\arg(\varphi(k)))g(t-k)$. By some efforts, one can see that $f_\varphi \in \mathcal{C}_\alpha$ and that $\|f_\varphi\|_{\mathcal{C}_\alpha} = 1$. Further, notice that $\mathrm{T}_{N,\varphi} f_\varphi = \sum_{k=-N}^{N} e^{\alpha|k|}|\varphi(k)|$ Thus, we have:

$$\|\mathrm{T}_{N,\varphi}\| = \sup_{\|f\|_{\mathcal{C}_\alpha}=1}|\mathrm{T}_{N,\varphi} f| \geqslant |\mathrm{T}_{N,\varphi} f_\varphi| = \sum_{k=-N}^{N} e^{\alpha|k|}|\varphi(k)|.$$

Now, assume that the statement in the Theorem for a suitable function φ is not correct. Then there exists a constant $C > 0$ (which depends on φ), for which it holds $\|T_{N,\varphi}\| \leqslant C$, $\forall N \in \mathbb{N}$, which in turn implies that:

$$|\varphi(k)| \leqslant Ce^{-\alpha|k|}, \quad \forall k \in \mathbb{Z} \tag{9}$$

Now, since $\hat{\varphi} \in \mathcal{S}$ (It is well-known that FT for $\mathcal{S}$ is a self-map), and in particular infinitely often differentiable, we can express $\hat{\varphi}$ by means of the uniformly convergent Fourier series $\hat{\varphi}(\omega) = \sum_{k=-\infty}^{+\infty}\varphi(k)e^{-i\omega k}$, $|\omega| \leqslant \pi$. Subsequently, consider the Laurent series:

$$G(z) := \underbrace{\sum_{k=-\infty}^{-1}\varphi(k)z^{-k}}_{=:G_1(z)} + \underbrace{\sum_{k=0}^{+\infty}\varphi(k)z^{-k}}_{=:G_2(z)}, \quad z \in \mathbb{C}\backslash\{0\}.$$

It is not hard to see that by (9) the power series G_1 converges and is holomorphic for $z < e^\alpha$, and as similar, the series G_2 for $z > e^{-\alpha}$. Altogether, both facts asserts that G converges in the annulus $e^{-\alpha} < |z| < e^\alpha$. Clearly, the unit circle is contained in the convergence region of G. Hence, we can imply that $G(e^{i\omega}) = \hat{\varphi}(\omega)$, $|\omega| \leqslant \pi$. Above equality, and the property of $\hat{\varphi}$ asserts, that $G(z) = 0$, for $z := e^{i\omega}$, where $\omega_g \leqslant |\omega| \leqslant \pi$. Previous observation, homomorphicity of G in the annulus $e^{-\alpha} < |z| < e^\alpha$, and the identity Theorem for holomorphic functions asserts that G is identical to zero on its convergence region, and in particular on the unit circle, which contradicts the fact that $\hat{\varphi} \not\equiv 0$, which is an implication of the assumption $\varphi \not\equiv 0$, since the FT is a bijection for $\mathcal{S}$. ■

Thus Theorem 4 asserts that the step (2) can not be founded rigorously in the sense of $\mathcal{S}^{'}$ for all functions in $\mathcal{C}_\alpha$, $\alpha > 0$, and of course for functions in $\mathcal{C}_0$, which negates the rigorosity of (3), and hence (1) in $\mathcal{S}^{'}$.

Remark 2: We are even able to show that above Theorem holds for the strictly smaller Banach space $\mathcal{C}_{\alpha,1}^{+}$, containing functions $f \in \mathcal{C}_\alpha$, for which $f(t) = 0$, $\forall t \leqslant 0$, and $\int_0^\infty |f(t)| \, \mathrm{d}t < \infty$ holds. The corresponding proof will be given in the preceding paper.

V. BLI for Larger Signal Classes

We have shown in this work, that the BLI of the form (1) is erroneous even for functions in $\mathcal{C}_0$ (cf. Theorems 2 and 3), which has relatively regularly behaviour. Furthermore, we have given a rigorous way to construct a BLI for functions in $\mathcal{PW}^1$ (cf. Remark 1). However, this signal class is strictly smaller than $\mathcal{C}_0$. This raises the question, whether it is possible to construct a BLI in any form for a signal space larger than $\mathcal{PW}^1$.

For $\mathcal{C}$, it is obvious that a BLI in our sense can not exists, since by Riemann-Lebesgue Lemma, functions in $\mathcal{PW}_\pi^1$ has to vanish at infinity. The question now is whether the situation changes, if one consider another notion of band-limitedness: A signal g band-limited to π, if its extension g to $\mathbb{C}$ fulfills g is analytic and:

$$\forall \epsilon > 0 : \exists C > 0 : \ |g(z)| \leqslant C e^{(\pi+\epsilon)|z|}, \ \forall z \in \mathbb{C}. \qquad (10)$$

Further, call the corresponding space $\mathcal{B}_\pi$. By this definition, we can define the BLI of a function f as a function f_π fulfilling:

$$f_\pi \in \mathcal{B}_\pi \quad f_\pi(k) = f(k), \quad \forall k \in \mathbb{Z} \qquad (11)$$

The following Theorem asserts that even the BLI for $\mathcal{C}$ can not exists in the sense (11):

Theorem 5: *Let $\alpha > 0$ be arbitrarily chosen. The set of all $f \in \mathcal{C}_\alpha$, for which a BLI f_π with band-limit π in the sense* (11) *exists, is a set of 1st category in $\mathcal{C}_\alpha$.*

Proof: Let be $f \in \mathcal{C}_\alpha$. A necessary condition for f to admit a BLI f_π in the sense (11) is certainly:

$$\limsup_{k\to\infty} |f(k)| \, e^{-\frac{\alpha}{2}|k|} < \infty, \qquad (12)$$

Now, we aim to show, that the set of $f \in \mathcal{C}_\alpha$, for which (12) holds, is a set of 1st category in $\mathcal{C}_\alpha$. Notice that this would immediately imply that the statement in the Theorem holds. To show this, consider for each $k \in \mathbb{Z}$, the continuous functional $\mathrm{T}_k : \mathcal{C}_\alpha \to \mathbb{C}$, given by $f \mapsto f(k)e^{-\frac{\alpha}{2}|k|}$. We can give an estimation $\|\mathrm{T}_k\| \geqslant e^{\frac{\alpha}{2}|k|}$, by means of the particular choice $f(t) = e^{\alpha|t|} \in \mathcal{C}_\alpha$, with $\|f\|_{\mathcal{C}_\alpha} = 1$. From there, it follows that $\sup_{k\in\mathbb{N}} \|\mathrm{T}_k\| = \infty$, and Theorem 1 asserts finally, that the set of $f \in \mathcal{C}_\alpha$, for which $\sup_{k\in\mathbb{N}} |\mathrm{T}_k f| = \infty$, or equivalently: for which (12) does not hold, is a residual set in $\mathcal{C}_\alpha$. For this reason, the set of all $f \in \mathcal{C}_\alpha$, for which (12) holds, as a complementary set of a residual set, is of 1st category in $\mathcal{C}_\alpha$, as desired. ■

In other words, Theorem 5 asserts that a BLI in any form can exists in the best case for functions in a topologically small subset of $\mathcal{C}_\alpha$. This abdicates clearly the existence of a BLI for the whole $\mathcal{C}_\alpha$, and correspondingly for the whole $\mathcal{C}$. Luckily, for the signal space $\mathcal{C}_0$ one can construct a BLI in the more general sense, cf. (11). In particular, it has a different form than that given in Remark 1: For an $f \in \mathcal{C}_0$, the following Valiron series converges pointwise and locally (but not globally) uniformly to a function in $\mathcal{B}_\pi$ [11]:

$$f_\pi(t) := f(t_0)\frac{\sin(\pi t)}{\sin(\pi t_0)} + (t - t_0) \sum_{k=-\infty}^{\infty} \frac{f(k)}{k-t_0}\frac{\sin(\pi(t-k))}{\pi(t-k)}, \ (13)$$

where $t_0 \in \mathbb{R}\backslash\mathbb{Z}$.

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