Positive Trigonometric Polynomials and One-Dimensional Discrete Phase Retrieval Problem

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***Abstract*—In this paper some results on Schur transform are reviewed to address the problem of one-dimensional discrete phase retrieval. The goal is to provide a test whether a sequence of input magnitude data gives a solution to one-dimensional discrete phase retrieval problem. It has been previously shown that this issue is related to the nonnegativity of trigonometric polynomials. The proposed method is similar to the table procedure for counting the multiplicities of zeros on unit circle. Examples and numerical results are also provided to indicate that the problem of one-dimensional discrete phase retrieval often does not have a solution.**

I. INTRODUCTION

In a number of different disciplines, including astronomy, wave-front sensing, x-ray crystallography, and holography, one encounters the phase-retrieval problem: Given the modulus of the Fourier transform of an object, reconstruct the object or, equivalently, reconstruct the Fourier phase [1]. This problem is broadly studied based on various types of *a priori* information about the underlying signal such as positivity and magnitude information on the signal [2]. Algorithms for various measurement schemes such as the method that exploits signal sparsity have been suggested [3]. In such cases the solution of phase retrieval problem is obtained based on supplementary information which may come from additional assumptions.

Positive trigonometric polynomials have been intensively used in many fields of signal processing such as spectral factorization and convex optimization [4]. Recently some results in the area bridge gaps between one-dimensional discrete phase retrieval problem and input magnitude data [5], [6]. It has been shown that the input magnitude data should satisfy certain conditions in order to provide the solution for one-dimensional discrete phase retrieval problem. These requirements ask for the corresponding trigonometric polynomial to be positive definite. Alternatively, given an arbitrary set of input magnitude data may not provide a solution. Instead of looking for additional assumptions or supplementary information to solve the one-dimensional discrete phase retrieval problem, the existence of the solution for the one-dimensional discrete phase retrieval problem is first analyzed in these approaches.

An essential step in order to prove the existence of the solution for the one-dimensional discrete phase retrieval problem is to determine whether a trigonometric polynomial is nonnegative or not. The test table procedure may be used to verify whether a polynomial is positive on unit circle [7], but this way is not very simple to implement. Basically this is a method for counting the multiplicities of zeros on unit circle [8]. A similar procedure involves computation of some determinants [9], which uses almost the same theoretical results, but in a different presentation.

In the following we shall present a way one can test the unit circle positivity of a polynomial, by using the Schur transform. This method is similar to the table procedure for counting the multiplicities of zeros on unit circle, but the form discussed in this work is useful for our goal. We shall recall few theoretical results related to this method. Examples and experimental results will be focused on one-dimensional discrete phase retrieval case.

The paper is organized as follows. First the one-dimensional discrete phase retrieval problem is briefly reviewed (Section II). Then some properties of nonnegative trigonometric polynomials are presented (Section III). Theoretical results useful to understand and follow the method based on Schur transform are recalled, and the testing procedure is described (Section IV). Examples (Section V) and numerical results (Section VI) are also provided.

A. Nomenclature

$\bar{z}$	complex conjugate of z
$X(z)$	z-transform of $x(n)$
$X(\omega)$	Fourier transform of $x(n)$
$X(k)$	DFT of $x(n)$
$\tilde{r}(n)$	circular autocorrelation of $x(n)$
$\hat{r}(n)$	the folded of circular autocorrelation
$\hat{S}(z)$	z-transform of $\hat{r}(n)$
$\hat{S}(\omega)$	Fourier transform of $\hat{r}(n)$
$\tilde{X}(k)$	input magnitude data

In the following polynomials will be considered as sum of powers of z^{-1}.

II. THE ONE DIMENSIONAL DISCRETE PHASE RETRIEVAL PROBLEM

Although certain constraints may be added according to application, the main one dimensional discrete phase retrieval problem can be stated as follows:

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Let $\tilde{X}(k)$, $k = 0, 1, \ldots, N-1$, be a sequence of N nonnegative numbers, which will be called the input magnitude data. To solve the one-dimensional discrete phase retrieval problem means to find $x(n)$ a discrete signal of finite length M ($M \leq N$) and for which its N-point Discrete Fourier Transform (DFT):

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, \ldots, N-1$$

satisfies

$$|X(k)| = \tilde{X}(k) \tag{1}$$

for all $k = 0, 1, \ldots, N-1$.

We have [10]:

Theorem 1: The one-dimensional discrete phase retrieval problem has a solution by method described below if and only if the trigonometric polynomial $\hat{S}(\omega) = \mathcal{F}\{\hat{r}(n)\}$ is nonnegative:

$$\hat{S}(\omega) \geq 0, \forall \omega \in [0, 2\pi]. \tag{2}$$

To obtain a direct solution to one dimensional discrete phase retrieval problem one may proceed as follows [6]:

1) Given $\tilde{X}(k)$ the input magnitude data;
2) Compute the circular autocorrelation:

$$\tilde{r}(n) = \frac{1}{2M-1} \sum_{k=0}^{2M-2} |\tilde{X}(k)|^2 e^{j\frac{2\pi kn}{2M-1}}; \tag{3}$$

3) Compute the folded circular autocorrelation $\hat{r}(n)$ using:

$$\hat{r}(n) = \begin{cases} \tilde{r}(n), & n = 0, 1, 2, \ldots, M-1 \\ \tilde{r}(2M-1+n), & n = -(M-1), \ldots, -1 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

4) Verify whether $\hat{S}(\omega) = \mathcal{F}\{\hat{r}(n)\}$ is a nonnegative trigonometric polynomial;
   a) If YES, we can proceed by solving the one dimensional discrete phase retrieval problem; one may use the zero allocation technique;
   b) If NOT, the one dimensional discrete phase retrieval problem does not have a solution.

III. About nonnegative trigonometric polynomials

From $\tilde{X}(k)$ one can obtain a closed form of $\hat{S}(z)$ and $\hat{S}(\omega)$ and we would like to verify whether $\hat{S}(\omega)$ is positive for all ω. To this end we can focus on $\hat{S}(z)$ to see whether $\hat{S}(z)$ is nonnegative for all z with $|z| = 1$, i.e. $\hat{S}(z)$ is nonnegative on unit circle.

Now we recall some properties of the polynomials which are nonnegative on the unit circle [4]:

1) Positive polynomials on unit circle are either symmetric or complex Hermitian or a positive constant;
2) Positive polynomials on unit circle contain two types of zeros:
   - zeros that are not on the unit circle and come in pairs z_k, $1/\overline{z}_k$;
   - zeros that are on the unit circle and have even multiplicity.

It follows that a standard technique for testing the unit circle positivity may consist of the following steps:

1) verify whether the polynomial $\hat{S}(z)$ is either symmetric or complex Hermitian or a positive constant;
2) count the number of zeros of the polynomial $\hat{S}(z)$ inside the unit disk;
3) count the multiplicities of zeros on unit circle of the polynomial $\hat{S}(z)$, if they exist;
4) in addition $\hat{S}(1)$ or $\hat{S}(-1)$ must be positive.

IV. The Schur transform and zeros multiplicities

The most popular tool to count the number of zeros inside the unit disk is based on the Schur transform.

For a polynomial of degree N in z^{-1} of the form [11]:

$$P(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N},$$

the reciprocal polynomial of P is defined by:

$$P^*(z) = \overline{a}_N + \overline{a}_{N-1} z^{-1} + \cdots + \overline{a}_0 z^{-N},$$

and its derivative is given by:

$$P'(z) = N a_0 + (N-1) a_1 z^{-1} + \cdots + a_{N-1} z^{-N+1}.$$

The Schur transform of the polynomial P of degree n is the polynomial TP of degree $N-1$ defined by

$$TP(z) = \overline{a}_0 P(z) - a_N P^*(z) = \sum_{k=0}^{N-1} (\overline{a}_0 a_k - a_N \overline{a}_{N-k}) z^{-k}. \tag{5}$$

The iterated Schur transforms T^2P, T^3P, ..., T^nP are defined by:

$$T^k P = T(T^{k-1}P), \quad k = 2, 3, \ldots, N. \tag{6}$$

We set $\gamma_k = T^k P(\infty)$, for $k = 1, 2, \ldots, N$.

We have the following results:

Theorem 2: **(Schur-Cohn Algorithm)** [11] Let P a polynomial of degree N in z^{-1}, $P \neq 0$. Then all zeros of P lie inside the closed unit disk $|z| < 1$ if and only if $\gamma_k > 0$, for all $k = 1, 2, \ldots, N$.

Under the additional hypothesis that all $\gamma_k \neq 0$, the algorithm can also be used to determine the exact number of zeros inside the unit disk [9], [11]. Let $\Gamma_k = \gamma_1 \gamma_2 \ldots \gamma_k$, for $k = 1, 2, \ldots, N$.

Theorem 3: [11] If for the polynomial $P(z)$, p of the products Γ_k are negative and the remaining $N-p$ are positive, then $P(z)$ has p zeros outside the unit disk, no zeros on the unit circle and $N-p$ zeros inside the unit disk.

Theorem 2 cannot be used if some γ_k are zero:

1) Such situation occurs when we have $T^{k+1}P = 0$.
2) It may happen also that $T^{k+1}P \neq 0$, but $\gamma_{k+1} = T^{k+1}P(\infty) = 0$.

To address these cases we have the following helpful definition [9]:

Definition 1: The polynomial $P(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}$ is said to be self-inversive if $a_k = \bar{a}_{N-k} u$, for all k, where $|u| = 1$.

The first case ($T^{k+1}P = 0$) appears generally whenever one of the polynomials $T^k P$ turns out to be self-reciprocal [11]. This is in fact the situation of one-dimensional discrete phase retrieval problem, when $\hat{S}(z)$ is self-reciprocal. To address this issue, we may compute the number of zeros inside the unit disk $\hat{S}(z)$ by focusing on its derivative of $\hat{S}'(z)$. Indeed, we have the following theorem:

Theorem 4: [9] If a polynomial is self-inversive, then it has as many zeros on the unit disk as the reciprocal of its derivative. That is, a self-reciprocal polynomial and its derivative have the same number of zeros outside the unit disk.

For the second case, we may consider the next result:

Theorem 5: [9] If the coefficients of the polynomial

$$P(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}$$

satisfy $a_k = \bar{a}_{N-k}|u|$, for $k = \overline{1, q-1}$, where $|u| = 1$ and $q \leq N/2$, then $P(z)$ has for $|z| \geq 1$ as many zeros as the polynomial

$$p_1(z) = \bar{B}_0 p(z) - B_{N+q} p^*(z),$$

where $b = (a_{N-q} - u\bar{a}_q)/a_N$ and

$$p(z) = (1 + 2b/|b| z^{-q}) P(z) = \sum_{j=0}^{N+q} B_j z^{-j}.$$

To find the number of zeros on unit circle, we shall apply the following result:

Theorem 6: [11] If for some $k < N$, $\Gamma_k \neq 0$, but $T^{k+1}P = 0$, then P has $N-k$ zeros on or symmetric in the unit circle at the zeros of $T^k P(z)$. If p of the Γ_j, $j = 1, 2, \ldots, k$ are negative, P has p additional zeros outside the unit circle and $k-p$ additional zeros inside the unit disk.

Using these results one can find the number of zeros of the polynomial $\hat{S}(z)$ inside and outside the unit circle. Consequently we can get the number of zeros on the unit circle. To find the multiplicity of zeros on the unit circle the resultant of the polynomial and its derivatives may be computed as well. Another way is to implement the Euclid's algorithm for the polynomial and its derivatives. The procedure is nicely described [7], [8] and will be taken into account in the following by using Schur transform for testing the existence of a solution for one-dimensional discrete phase retrieval problem.

V. Example

In the following we shall illustrate the use of Theorems 2 to 6 to test the positivity on unit circle of polynomial, and in this way to determine the existence of a solution for one-dimensional discrete phase retrieval problem.

Let $N = 5$ and

$$\tilde{X}(k) = \begin{cases} 1+\alpha, & k = 0; \\ 1, & k = 1, 2, 3, 4. \end{cases} \quad (7)$$

For different α, we get solution or no solution for the one-dimensional discrete phase retrieval problem [5], [6].

Then

$$\tilde{r}(n) = \mathrm{DFT}^{-1}\{|\tilde{X}(k)|^2\} = \begin{cases} 1 + \frac{1}{5}(\alpha^2 + 2\alpha), & n = 0; \\ \frac{1}{5}(\alpha^2 + 2\alpha), & n = \overline{1, 4}. \end{cases}$$

We have

$$\hat{r}(n) = \begin{cases} 1 + \frac{1}{5}(\alpha^2 + 2\alpha), & n = 0; \\ \frac{1}{5}(\alpha^2 + 2\alpha), & n = \pm 1, \pm 2. \end{cases} \quad (8)$$

Thus $\hat{S}(z)$ corresponding to (7) is given by:

$$\begin{aligned} \hat{S}(z) &= \mathcal{Z}\{\hat{r}(n)\} = 1 + \tfrac{1}{5}(\alpha^2 + 2\alpha) + \\ &\tfrac{2}{5}(\alpha^2 + 2\alpha)\tfrac{z + z^{-1}}{2} + \tfrac{2}{5}(\alpha^2 + 2\alpha)\tfrac{z^2 + z^{-2}}{2} = \\ &= \tfrac{1}{5}(\alpha^2 + 2\alpha)\left[z^2 + z^{-2} + z + z^{-1} + \tfrac{\alpha^2 + 2\alpha + 5}{\alpha^2 + 2\alpha}\right]. \end{aligned} \quad (9)$$

For the beginning, let us consider the polynomial:

$$\hat{P}(z) = \frac{5\hat{S}(z)}{\alpha^2 + 2\alpha} = z^2 + z^{-2} + z + z^{-1} + \frac{\alpha^2 + 2\alpha + 5}{\alpha^2 + 2\alpha} = z^2 + z + \beta + z^{-1} + z^{-2},$$

where

$$\beta = \frac{\alpha^2 + 2\alpha + 5}{\alpha^2 + 2\alpha}.$$

To have $\hat{P}(-1) > 0$, we need $\beta > 0$.

To test positivity on unit circle of $\hat{S}(z)$ given by (9), we shall focus on the zeros of the polynomial

$$P(z) = 1 + z^{-1} + \beta z^{-2} + z^{-3} + z^{-4}$$

to see whether $P(z)$ has zeros with even or odd multiplicity on unit circle. As we expected, $P(z)$ is reciprocal and the Schur transform has to be applied for its reciprocal derivative

$$\tilde{P}'(z) = 4 + 3z^{-1} + 2\beta z^{-2} + z^{-3}.$$

We get:

$$T\tilde{P}'(z) = (8\beta - 3)z^{-2} + (12 - 2\beta)z^{-1} + 15;$$

$$T^2\tilde{P}'(z) = [180 - 30\beta - (8\beta - 3)(12 - 2\beta)]z^{-1} + 225 - (8\beta - 3)^2 = 4(4\beta - 9)[(\beta - 6)z^{-1} - 2(3 + 2\beta)];$$

$$T^3\tilde{P}'(z) = [225 - (8\beta - 3)^2]^2 - [180 - 30\beta - (8\beta - 3)(12 - 2\beta)]^2;$$

$$\gamma_1 = 15; \quad \gamma_2 = 225 - (8\beta - 3)^2 = 8(9 - 4\beta)(3 + 2\beta);$$

$$\gamma_3 = [225 - (8\beta - 3)^2]^2 - [180 - 30\beta - (8\beta - 3)(12 - 2\beta)]^2 = 240\beta(\beta + 4)(4\beta - 9)^2;$$

$$\Gamma_1 = 15; \quad \Gamma_2 = -120(4\beta - 9)(3 + 2\beta);$$

$$\Gamma_3 = -28800(3 + 2\beta)\beta(\beta + 4)(4\beta - 9)^3.$$

It follows that:

1) When $\beta \in (0, 9/4)$, $\tilde{P}'(z)$ has 3 zeros inside the unit disk and no one outside the unit disk. Thus all $P(z)$ has no zeros outside the unit disk. Since it is reciprocal, it has no zeros inside the unit disk. It follows that $P(z)$ has all zeros on the unit circle and all zeros are single (otherwise some γ_k of $\tilde{P}'(z)$ should be zero).
2) When $\beta > 9/4$, then $\tilde{P}'(z)$ has two zeros outside the unit disk, thus $P(z)$ has two zeros outside the unit disk and two zeros inside the unit disk. Consequently $P(z)$ has no zeros on the unit circle. Thus $P(z)$ is positive for all $|z| = 1$.
3) When $\beta = 9/4$, we have $\Gamma_1 \neq 0$ and $T^2\tilde{P}'(z) = 0$, thus for $T^1\tilde{P}'(z) = 15 + 15/2z^{-1} + 15z^{-2}$ we have to consider the derivative of $g(z) = 1 + 1/2z^{-1} + z^{-2}$. We have $g'(z) = 1/2 + 2z^{-1}$ which has one zero inside the unit circle. Since $g'(z)$ has no zeros outside the unit circle, then $g(z)$ has no zeros inside and outside the unit circle. Thus $T^1\tilde{P}'(z)$ has 2 zeros on the unit circle ($z_{1,2} = -0.25 \pm j0.9682$), consequently $P'(z)$ has two zeros on unit circle, ($z_{1,2} = -0.25 \pm j0.9682$), one zero inside the unit disk ($z_3 = -0.25$) and no zeros outside the unit disk. It means that $P(z)$ has no zeros outside the unit disk and it has all the zeros on the unit disk. It can be easily verified that these zeros have double multiplicity and they are the zeros of $P'(z)$.

Thus P(z) is nonnegative if $\beta \geq 9/4$.

To conclude this section, the Schur transform approach is difficult to utilize especially when we have to deal with polynomials having multiple zeros. On the other hand, the Schur transform helps us to verify the existence of a solution for one-dimensional discrete phase retrieval problem.

The example discussed in this section suggests that in many cases the input magnitude data may not provide a solution to this problem. Moreover, the average and sometimes the variation of the input magnitude data may affect the existence of a solution to the one-dimensional discrete phase retrieval problem.

VI. Experimental results

In the following we shall present the results of two sets of simulations. The first set of simulations looks for the percentage of polynomials having zeros on unit circle. The second set of simulations considers polynomials obtained from input magnitude data and we shall verify whether they are nonnegative.

A. First set of simulations

We have generated 10^4 different polynomials for everyone of the following classes:

1) polynomials with complex coefficients, where the real part and the imaginary part of coefficients are uniform distributed between [-0.5,0.5];
2) polynomials with complex coefficients, where the real part and the imaginary part of coefficients are normal distributed with zero mean and unit variance;

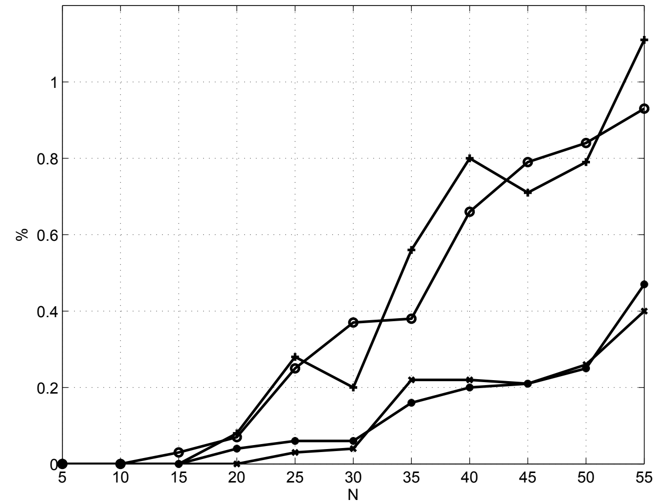

Fig. 1. Percentage of polynomials having zeros on unit circle: polynomials with complex uniform distributed coefficients (+−), polynomials with complex Gaussian distributed coefficients (○−), derivatives of reciprocal polynomials with complex uniform distributed coefficients ×−) and derivatives of reciprocal polynomials with complex Gaussian distributed coefficients (∗−).

3) derivative of reciprocal polynomials, where the reciprocal polynomials coefficients are uniform distributed between [-0.5,0.5];
4) derivative of reciprocal polynomials, where the reciprocal polynomials coefficients are normal distributed with zero mean and unit variance.

Then we have computed the Schur transform for every polynomial and after that we evaluate the percentage of polynomials having zeros on unit circle. The outcomes are presented in Figure 1 for every class. One can see that the polynomials having zeros on unit circle are very seldom for these four classes.

We recall that for a reciprocal polynomial the Schur transform is zero, and to find the zeros on unit circle of a reciprocal polynomial we have to apply the Schur transform to its derivative (Section IV).

When computing the Schur transform it may happen that the coefficients decrease very rapidly or sometimes increases quite fast. Since only the sign of $\gamma_k = T^k P(\infty)$ has importance for our goal, we scale the coefficients; actually we proceed as it is recommended in [8].

To conclude this experiment, the polynomials having zeros on unit circle are very seldom. For our goal it is important to note that reciprocal polynomials have seldom multiple zeros on unit circle.

B. Second set of simulations

First we have generated 10^4 different input magnitude data, every set being uniform distributed between [0;1], [0.5;1.5] and [1;2] respectively. For every input magnitude data, we have computed the circular autocorrelation, the folded of circular autocorrelation and its z-transform $\hat{S}(z)$. Then we have calculated the Schur transform of the derivative of $\hat{S}(z)$.

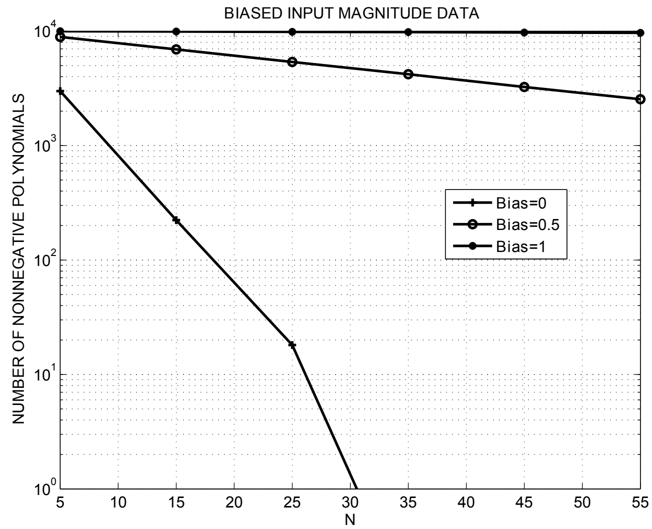

Fig. 2. Number of nonnegative polynomials obtained from input magnitude data with uniform distribution between [0;1] ($+-$), [0.5;1.5] $\circ-$ and [1;2] ($*-$) respectively.

In case that the Schur transform of $\hat{S}'(z)$ is identically zero for certain $(k+1)$, we restart the process with the derivative of the last nonzero $T^k(\hat{S}'(z))$. Using this approach we have found the number of nonnegative polynomials obtained from these input magnitude data distributions. The outcomes are presented in Figure 2.

Our remarks are as follows:

- If the mean of input magnitude data is rather close to zero, there is likely to get an $\hat{S}(z)$ that is negative. Since usually the distribution of input magnitude data is biased towards zero, we can conclude that for large N the discrete phase retrieval has very seldom solution.
- If the input magnitude data is severely biased towards positive values, then almost all $\hat{S}(z)$ is nonnegative. In such case one can find often a solution to one-dimensional discrete phase retrieval.

Secondly we have generated 10^4 different input magnitude data:

- the first class (I) has data obtained from modulus of a Gaussian distribution with mean zero and unit variance;
- the second class (II) has data obtained from modulus of a Gaussian distribution with mean zero and its standard deviation is 5;
- the third class (III) has data obtained from modulus of a Gaussian distribution with mean zero and unit variance and then biased by 3.

We have proceeded as previously and the outcomes are shown in Table I. For these distributions we performed other series of simulations, and the behavior was almost the same, with very seldom nonnegative polynomials $\hat{S}(z)$ for large N.

To conclude the one-dimensional discrete phase retrieval is sensitive to the input magnitude data and we may have no solution to this problem. Moreover, when the mean of input magnitude data is small, the existence of a solution for one-dimensional discrete phase retrieval problem is not guaranteed.

N	5	15	25
Set I	1492	25	0
Set II	1511	33	1
Set III	1505	34	0

TABLE I
NUMBER OF NONNEGATIVE POLYNOMIALS OBTAINED FROM INPUT MAGNITUDE DATA WITH POSITIVE GAUSSIAN DISTRIBUTION.

VII. CONCLUSION

In this paper some results on Schur transform have been reviewed to address the problem of one-dimensional discrete phase retrieval. The goal was to test whether an arbitrary sequence of input magnitude data may provide a solution to one-dimensional discrete phase retrieval problem.

Simulations show that we cannot have always a solution. For certain distributions of input magnitude data, which are common used in measurements, we can expect some regular behavior. This indicates that further work is needed to characterize these distributions of sequences of input magnitude data.

To conclude our achievements, before trying to solve one dimensional discrete phase retrieval problem, it is recommended to compute the Schur transform approach of corresponding trigonometric polynomial and to verify whether it is nonnegative. Although this requires a supplementary work, it may be useful since the one-dimensional discrete phase retrieval problem may have no solution.

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