Signal and System Spaces with Non-Convergent Sampling Representation

Holger Boche and Ullrich J. Mönich
Technische Universität München
Lehrstuhl für Theoretische Informationstechnik

***Abstract*—The approximation of linear time-invariant (LTI) systems by sampling series is a central task of signal processing. For the Paley–Wiener space $\mathcal{PW}_\pi^1$ of bandlimited signals with absolutely integrable Fourier transform, it is known that there exist signals and stable LTI systems such that the canonical approximation process diverges. In this paper we analyze the structure of the sets of signals and systems creating divergence and show that both sets are jointly spaceable, i.e., contain subsets such that every linear combination of signals and systems from these subsets, which is not the zero element, leads to divergence. In signal processing applications the linear structure of the involved signal spaces is essential. Here, we show that the same linear structure also holds for the sets of signals and systems creating divergence.**

I. Motivation and Introduction

A central problem in signal processing is the approximation of linear time-invariant (LTI) systems, like the Hilbert transform or the derivative, by sampling series [1], [2], [3], [4], [5], [6]. For a given bandlimited input signal f and stable LTI system T, the canonical approximation process is given by

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k), \tag{1}$$

where $h_T = T \operatorname{sinc}$ denotes the response of the system T to the sinc-function [4, Sec. 4.4]. The convergence of (1) is not guaranteed and has to be checked from case to case.

In [7], [8] the convergence behavior of (1) was analyzed for signals in the Paley–Wiener space $\mathcal{PW}_\pi^1$ of bandlimited signals with absolutely integrable Fourier transform. It was shown that for each $t \in \mathbb{R}$ there exists a stable LTI system T and a signal $f \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = \infty, \tag{2}$$

i.e., that the approximation error grows arbitrarily large. However, it is not clear what structure the sets of signals and systems creating divergence have. In this paper we will address this question.

A second approximation process, closely related to (1), is the mixed-signal representation

$$\sum_{k=-\infty}^{\infty} f(t-k) h_T(k). \tag{3}$$

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Formally, (3) differs from (1) only in the position of the time variable t. However, this difference has a significant impact on how the approximation process can be implemented. In (3), for a fixed $t \in \mathbb{R}$, we need the signal values on the discrete grid $\{t-k\}_{k\in\mathbb{Z}}$ in order to calculate the system approximation $(Tf)(t)$. For different $t \in \mathbb{R}$ we need other signal values. As t ranges over $[0,1]$ we need all the signal values $f(\tau)$, $\tau \in \mathbb{R}$. In the mixed signal representation the signal f is processed in its analog form, and the necessary operations like delay, multiplication, and addition, which are typical for digital circuits, need to be implemented for analog quantities. As in the case of the approximation process (1), the mixed-signal process (3) is only useful if we have convergence. In Section IV we will see that (1) and (3) have exactly the same convergence behavior for the considered signal space. Hence, our result for the approximation process (1) is equally true for the mixed-signal process (3).

A signal space in signal processing is characterized, among other properties, by its linearity. That is, adding two signals from the signal space or multiplying a signal from the signal space with a scalar gives again a signal in the signal space. The same linear structure also holds for the set of stable LTI systems. This is important for signal processing applications, because it allows to compose complex signals and systems out of elementary and simple building blocks. In addition to the linear structure, it is also useful to introduce a norm on the spaces in order to compare the elements of the spaces, i.e., signals and systems.

In this paper we study the structure of the sets of signals and systems creating divergence. Due to the importance of the signal space structure in signal processing, it would be interesting to know whether these sets contain subsets which are signal spaces in the sense that they exhibit a linear structure. In this case any linear combination of signals or systems from those subsets, which is not the zero element, would lead to divergence as well.

Note that it is significantly more difficult to show a linear structure in the set of signals and systems with divergent system approximation process, compared to showing a linear structure in the set of signals and systems with convergent system approximation process. If we have two signals f_1 and f_2, for which (1) converges, it is clear that the sum of both signals, i.e., f_1+f_2, is a signal for which we have convergence as well. Hence, any finite linear combination of signals with

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convergent system approximation process will be a signal with convergent system approximation process. However, for divergence this is not true. Given two signals w_1 and w_2 for which (1) diverges, we cannot conclude that the sum of both signals, i.e., $w_1 + w_2$, is a signal for which (1) diverges. This can be easily seen by choosing $w_1 = f_1 + g$ and $w_2 = f_1 - g$, where f_1 is any signal with convergent system approximation process and g any signal with divergent system approximation process. Obviously, for the sum $w_1 + w_2 = 2f_1$ we do not have divergence. This shows that the sum of two signals, each of which leads to divergence, does not necessarily lead to divergence.

In Section IV we will prove that the sets of signals and systems creating divergence are spaceable, i.e., contain a closed infinite dimensional subspace with linear structure. We will even show that both sets are jointly spaceable in the sense that there exist two closed infinite dimensional subspaces D_{sig} and D_{sys}, such that for all pairs of signals and systems $(f, T) \in D_{\text{sig}} \times D_{\text{sys}}$, $f \not\equiv 0$, $T \not\equiv 0$, we have divergence as stated in (2).

II. General Notation

Let $\hat{f}$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. $L^p(\mathbb{R})$, $1 \leq p < \infty$, is the space of all measurable, pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. $L^p[t_1, t_2]$, $1 \leq p < \infty$, $\sigma > 0$, is the space of all measurable, pth-power Lebesgue integrable functions on $[t_1, t_2]$. $C[t_1, t_2]$ denotes the space of all continuous functions on $[t_1, t_2]$. For $1 \leq p \leq \infty$, $\mathcal{PW}_\pi^p$ denotes the Paley-Wiener space of functions f with a representation $f(z) = 1/(2\pi) \int_{-\pi}^{\pi} g(\omega)\, \mathrm{e}^{iz\omega}\ \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\pi, \pi]$. If $f \in \mathcal{PW}_\pi^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\pi^p$, $1 \leq p < \infty$, is given by $\|f\|_{\mathcal{PW}_\pi^p} = (1/(2\pi) \int_{-\pi}^{\pi} |\hat{f}(\omega)|^p\ \mathrm{d}\omega)^{1/p}$.

We briefly review some definitions and facts about stable linear time-invariant (LTI) systems, which will be relevant. A linear system $T : \mathcal{PW}_\pi^p \to \mathcal{PW}_\pi^p$, $1 \leq p \leq \infty$, is called stable if the operator T is bounded, i.e., if $\|T\| := \sup_{\|f\|_{\mathcal{PW}_\pi^p} \leq 1} \|Tf\|_{\mathcal{PW}_\pi^p} < \infty$. Furthermore, it is called time-invariant if $(Tf(\,\cdot - a))(t) = (Tf)(t - a)$ for all $f \in \mathcal{PW}_\pi^p$ and $t, a \in \mathbb{R}$.

In this paper we are mainly interested in stable LTI systems operating on the space $\mathcal{PW}_\pi^1$, i.e., in the case $p = 1$. By $\mathcal{T}$ we denote the set of stable LTI systems $T \colon \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$, and by $\mathcal{T}_{\mathrm{C}}$ the set of systems $T \in \mathcal{T}$ with continuous $\hat{h}_T$. The operator norm of a stable LTI system T is given by $\|T\| = \|\hat{h}_T\|_{L^\infty[-\pi,\pi]}$. For every stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ there exists exactly one function $\hat{h}_T \in L^\infty[-\pi, \pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_T(\omega)\, \mathrm{e}^{i\omega t}\ \mathrm{d}\omega, \quad t \in \mathbb{R}, \tag{4}$$

for all $f \in \mathcal{PW}_\pi^1$. Conversely, every function $\hat{h}_T \in L^\infty[-\pi, \pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$. Hence, we can identify stable LTI systems with $L^\infty[-\pi, \pi]$ functions. By $Q \colon \mathcal{T} \to L^\infty[-\pi, \pi]$ we denote the isometric isomorphism that performs this mapping. We have $h_T = T \operatorname{sinc}$, where sinc denotes the usual sinc-function which is defined by $\operatorname{sinc}(t) = \sin(\pi t)/(\pi t)$ for $t \neq 0$ and $\operatorname{sinc}(t) = 1$ for $t = 0$.

Remark 1. Adding two signals in $\mathcal{PW}_\pi^1$ or multiplying a signal in $\mathcal{PW}_\pi^1$ with a scalar gives again a signal in $\mathcal{PW}_\pi^1$. That is $\mathcal{PW}_\pi^1$ is a signal space with linear structure. The same is true for the space of stable LTI systems $\mathcal{T}$.

III. Spaceability

In the introduction we discussed the importance of linear structures in signal spaces. For the sets of signals and systems with convergent approximation process it is easy to show the linearity. However, for the sets of signals and systems with divergent approximation process, this is not the case, and an analysis is much more intricate. In the following we want to answer the question whether the sets of signals and systems with divergent approximation process contain infinite dimensional subspaces, i.e., sets with linear structure.

Lineability and spaceability are two appropriate mathematical concepts to study the existence of linear structures in general sets. A subset S of a Banach space X is said to be lineable if $S \cup \{0\}$ contains an infinite dimensional subspace. A subset S of a Banach space X is said to be spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace of X.

Remark 2. Spaceability is a stronger property than lineability. Every spaceable set is lineable but not vice-versa.

Both concepts were recently introduced and have been used for example in [9], [10], [11], [12], In [9] it was proved that the set of continuous nowhere differentiable functions on $\mathbb{R}$ is lineable. Later, it was shown that the set of continuous nowhere differentiable functions on $C[0, 1]$ is spaceable [10]. The divergence of Fourier series was analyzed in [12], where it was shown that the set of functions in $L^1(\partial\mathbb{D})$, whose Fourier series diverges everywhere on $\partial\mathbb{D}$ is spaceable. Spaceability and lineability in different setting was further analyzed in [13], [14], [15], [16]. In [16] spaceability of certain mappings between sequence spaces was studied, and in [14] spaceability of Banach spaces of sequences. Lineability of linear spaces was considered in [15].

IV. Main Result

Theorem 1. *There exist an infinite dimensional closed subspace $D_{sig} \subset \mathcal{PW}_\pi^1$ and an infinite dimensional closed subspace $D_{sys} \subset \mathcal{T}_{\mathrm{C}}$ such that for all $f \in D_{sig}$, $f \not\equiv 0$, and all $T \in D_{sys}$, $T \not\equiv 0$, we have*

$$\limsup_{N \to \infty} \left| \sum_{k=-N}^{N} f(k) h_T(-k) \right| = \infty.$$

Theorem 1 shows that there exist a spaceable set of signals $D_{\text{sig}} \subset \mathcal{PW}_\pi^1$ and a spaceable set of stable LTI systems $D_{\text{sys}} \subset \mathcal{T}_{\mathrm{C}} \subset \mathcal{T}$ such that the system approximation process (1) diverges at $t = 0$ for any pair of signal and system

$(f, T) \in D_{\text{sig}} \times D_{\text{sys}}$, $f \not\equiv 0$, $T \not\equiv 0$, chosen from the two sets. In the previous expression, we denoted the zero element by 0. For the signal space it is the signal f that is identically zero, i.e., $f(t) = 0$ for all $t \in \mathbb{R}$, and for the system space it is the LTI system T with $\hat{h}_T(\omega) = 0$ for almost all $\omega \in [-\pi, \pi]$. From the context it will be always clear which zero element we refer to when writing 0.

Clearly, joint spaceability implies ordinary spaceability. Further, we have the following two conjectures.

Conjecture 1. *Let $T \in \mathcal{T}$ be a stable LTI system. If there exists a signal $f \in \mathcal{PW}_\pi^1$ with*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(-k) \right| = \infty, \tag{5}$$

then the set of signals $f \in \mathcal{PW}_\pi^1$ with (5) *is spaceable.*

Conjecture 2. *Let $f \in \mathcal{PW}_\pi^1$. If there exists a stable LTI system $T \in \mathcal{T}$ with*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(-k) \right| = \infty, \tag{6}$$

then the set of stable LTI systems $T \in \mathcal{T}$ with (6) *is spaceable.*

Remark 3. Theorem 1 is concerned with the sets of signals and systems for which we have divergence. As for convergence, we have the following well-known situation. For all signals $f \in \mathcal{PW}_\pi^2 \subset \mathcal{PW}_\pi^1$ and all systems $T \in \mathcal{T}$ we have

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| (Tf)(t) - \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = 0,$$

i.e. we have lineability of the set of input signals which lead to a convergent system approximation for all stable LTI systems. Further, for all stable LTI FIR-systems T_{FIR}, i.e., systems $T \in \mathcal{T}$ with $h_T(k) \neq 0$ for only finitely many $k \in \mathbb{Z}$, we have for all $f \in \mathcal{PW}_\pi^1$ that

$$(Tf)(0) = \sum_{k=-\infty}^{\infty} f(k) h_{T_{\text{FIR}}}(-k),$$

because only finitely many summands are non-zero. Therefore, we also have lineability of the set of systems for which $(Tf)(0)$ can be represented by a finite sampling series for all signals in $\mathcal{PW}_\pi^1$.

The divergence of the approximation process (1) for arbitrary $t \neq 0$ follows easily from Theorem 1 and is stated in the following corollary, the proof of which is given after the proof of Theorem 1.

Corollary 1. *Let $t \in \mathbb{R}$ be arbitrary but fixed. There exist an infinite dimensional closed subspace $D_{sig} \subset \mathcal{PW}_\pi^1$ and an infinite dimensional closed subspace $D_{sys2} \subset \mathcal{T}_C$ such that for all $f \in D_{sig}$, $f \not\equiv 0$, and all $T \in D_{sys2}$, $T \not\equiv 0$, we have*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = \infty.$$

We have the same result for the mixed-signal representation.

Corollary 2. *Let $t \in \mathbb{R}$ be arbitrary but fixed. There exist an infinite dimensional closed subspace $D_{sig2} \subset \mathcal{PW}_\pi^1$ and an infinite dimensional closed subspace $D_{sys} \subset \mathcal{T}_C$ such that for all $f \in D_{sig2}$, $f \not\equiv 0$, and all $T \in D_{sys}$, $T \not\equiv 0$, we have*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(t-k) h_T(k) \right| = \infty.$$

For the proof of Theorem 1 we need the following lemma, the proof of which is omitted due to space constraints.

Lemma 1. *There exist two sequences of functions $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ with:*

1) *The functions ϕ_n, $n \in \mathbb{N}$, are finitely linearly independent, $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{PW}_\pi^1$, and there exists a constant C_1 such that $\|\phi_n\|_{\mathcal{PW}_\pi^1} \leq C_1$ for all $n \in \mathbb{N}$.*
2) *The functions g_n, $n \in \mathbb{N}$, are finitely linearly independent, $\{\hat{g}_n\}_{n\in\mathbb{N}} \subset C[-\pi, \pi]$, and there exists a constant C_2 such that $\|\hat{g}_n\|_\infty \leq C_2$ for all $n \in \mathbb{N}$. Further, we have $g_n(k) = 0$ for all negative integers k and all $n \in \mathbb{N}$.*
3) *For all $n, m \in \mathbb{N}$ there exists a sequences $\{N_r(n,m)\}_{r\in\mathbb{N}}$ and a constant C_3 such that*

$$\limsup_{r\to\infty} \left| \sum_{k=0}^{N_r(n,m)} \phi_n(-k) g_m(k) \right| = \infty$$

and

$$\sup_{r\in\mathbb{N}} \left| \sum_{k=0}^{N_r(\hat{n},\hat{m})} \phi_n(-k) g_m(k) \right| \leq C_3$$

for all $\hat{n}, \hat{m} \in \mathbb{N}$ with $(\hat{n}, \hat{m}) \neq (n, m)$.

Now we are in the position to prove Theorem 1.

Sketch of the proof of Theorem 1. Let $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be the two sequences of functions from Lemma 1. For $n \in \mathbb{N}$ and $t \in \mathbb{R}$ let

$$\xi_n^{(1)}(t) = \frac{1}{2^n 2C_1} \phi_n(t),$$

$$h_n^{(1)}(t) = \frac{1}{2^n 2C_2} g_n(t),$$

and

$$e_n(t) = \frac{\sin(\pi(t - 2^n))}{\pi(t - 2^n)}.$$

According to Paley's theorem [17, p. 104], $\{e_n\}_{n\in\mathbb{N}}$ is a basic sequence[1] in $\mathcal{PW}_\pi^1$. Further, $\{\hat{e}_n\}_{n\in\mathbb{N}}$ is a basic sequence in $L^\infty[-\pi, \pi]$ [18, p. 247]. Now we consider

$$\xi_n(t) = \xi_n^{(1)}(t) + e_n(t)$$

and

$$h_n(t) = h_n^{(1)}(t) + e_n(t).$$

[1]A sequence $\{f_n\}_{n\in\mathbb{N}}$ in a Banach space X is a basic sequence in X if it is a basis for its closed linear span.

We have $\|e_n^*\|_{\mathcal{PW}_\pi^\infty} = 1$ and $\|\hat{e}_n^*\|_{L^1[-\pi,\pi]} = 1$. Thus, it follows that

$$\sum_{n=1}^{\infty} \|e_n^*\|_{\mathcal{PW}_\pi^\infty} \|\xi_n - e_n\|_{\mathcal{PW}_\pi^1} = \frac{1}{2} < 1$$

and

$$\sum_{n=1}^{\infty} \|\hat{e}_n^*\|_{L^1[-\pi,\pi]} \|\hat{h}_n - \hat{e}_n\|_{C[-\pi,\pi]} = \frac{1}{2} < 1.$$

Hence, $\{\xi_n\}_{n\in\mathbb{N}}$ is a basic sequence for $\mathcal{PW}_\pi^1$ that is equivalent to $\{e_n\}_{n\in\mathbb{N}}$, and $\{\hat{h}_n\}_{n\in\mathbb{N}}$ is a basic sequence for $L^\infty[-\pi,\pi]$ that is equivalent to $\{\hat{e}_n\}_{n\in\mathbb{N}}$ [19, p. 46]. Further, there exists a constant $C_4 > 0$ such that

$$C_4 \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{n=1}^{\infty} a_n e_n \right\|_{\mathcal{PW}_\pi^1} \leq \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}.$$

Let D_{sig} denote the closure in the $\mathcal{PW}_\pi^1$-norm of the set

$$\left\{ \sum_{n=1}^{M} a_n \xi_n \colon a_n \in \mathbb{R}, M \in \mathbb{N} \right\}.$$

We have $f \in D_{\text{sig}}$ if and only if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. For every $f \in D_{\text{sig}}$ there exists a unique l^2-sequence $\{a_n\}_{n\in\mathbb{N}}$ such that

$$f = \sum_{n=1}^{\infty} a_n \xi_n.$$

Further let $\hat{D}_{\text{sys1}}$ denote the closure in the $C[-\pi,\pi]$-norm of the set

$$\left\{ \sum_{n=1}^{M} b_n \hat{h}_n \colon b_n \in \mathbb{R}, M \in \mathbb{N} \right\}.$$

We have $\hat{h} \in \hat{D}_{\text{sys1}}$ if and only if $\hat{h}$ has a coefficient sequence $\{b_n\}_{n\in\mathbb{N}}$ with $\sum_{n=1}^{\infty} |b_n| < \infty$. The coefficient sequence defines $\hat{h}$ uniquely. Clearly, every $\hat{h}$ uniquely defines a stable LTI system $T = Q^{-1}\hat{h}$. We denote the corresponding space of LTI systems by $D_{\text{sys1}} = Q^{-1}\hat{D}_{\text{sys1}}$.

Let $f \in D_{\text{sig}}$, $f \not\equiv 0$, and $\hat{h} \in \hat{D}_{\text{sys1}}$, $\hat{h} \not\equiv 0$, both be arbitrary but fixed. Then we have the expansions

$$f(t) = \sum_{n=1}^{\infty} a_n(f) \xi_n(t), \quad t \in \mathbb{R},$$

and

$$\hat{h}(\omega) = \sum_{n=1}^{\infty} b_n(h) \hat{h}_n(\omega), \quad \omega \in [-\pi,\pi].$$

Let n_0 denote the smallest natural number n such that $|a_n(f)| > 0$, and m_0 the smallest natural number m such that $|b_m(h)| > 0$. Clearly, we have

$$\begin{aligned} f(t) &= \sum_{n=n_0}^{\infty} a_n(f) \xi_n(t) \\ &= \underbrace{\sum_{n=n_0}^{\infty} a_n(f) e_n(t)}_{=A(t)} + \underbrace{\sum_{n=n_0}^{\infty} a_n(f) \xi_n^{(1)}(t)}_{=F_1(t)} \end{aligned}$$

and

$$\begin{aligned} h(t) &= \sum_{m=m_0}^{\infty} b_m(h) h_m(t) \\ &= \underbrace{\sum_{m=m_0}^{\infty} b_m(h) e_m(t)}_{=B(t)} + \underbrace{\sum_{m=m_0}^{\infty} b_m(h) h_m^{(1)}(t)}_{=G_1(t)}. \end{aligned}$$

The next step of the proof is to consider

$$\begin{aligned} \sum_{k=0}^{N} f(k)h(k) &= \sum_{k=0}^{N} A(k)B(k) + \sum_{k=0}^{N} A(k)G_1(k) \\ &\quad + \sum_{k=0}^{N} B(k)F_1(k) + \sum_{k=0}^{N} G_1(k)F_1(k) \end{aligned}$$

for $N \in \mathbb{N}$. Using the properties 1) – 3) from Lemma 1, it can be shown that the absolute values of the first, second, and third summands are bounded above by some constant independently of N, and that for the fourth summand we have

$$\limsup_{N\to\infty} \left| \sum_{k=0}^{N} G_1(k) F_1(k) \right| = \infty.$$

Since $h(k) = 0$ for $k < 0$, this implies that

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h(k) \right| = \infty.$$

To complete the proof, we consider the space $D_{\text{sys}} = Q^{-1}RQD_{\text{sys1}}$, where $R \colon f \mapsto f(-\,\cdot\,)$ denotes the time-reversal operator. D_{sys} is an infinite dimensional closed subspace of $\mathcal{T}_{\text{C}}$ and we have

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(-k) \right| = \infty$$

for all $f \in D_{\text{sig}}$, $f \not\equiv 0$, and all $T \in D_{\text{sys}}$, $T \not\equiv 0$. □

Proof of Corollary 1. From Theorem 1 we know that there exist an infinite dimensional closed subspace $D_{\text{sig}} \subset \mathcal{PW}_\pi^1$ and an infinite dimensional closed subspace $D_{\text{sys}} \subset \mathcal{T}_{\text{C}}$, such that for all $f \in D_{\text{sig}}$, $f \not\equiv 0$, and all $T \in D_{\text{sys}}$, $T \not\equiv 0$, we have

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(-k) \right| = \infty. \tag{7}$$

Let $t \in \mathbb{R}$ be arbitrary but fixed, and consider the operator $U \colon L^\infty[-\pi,\pi] \to L^\infty[-\pi,\pi], \hat{h}_T \mapsto \hat{h}_T \,\mathrm{e}^{-i\,\cdot\,t}$. U is a bounded, linear, and invertible operator with bounded inverse. Hence, $D_{\text{sys2}} = Q^{-1}UQD_{\text{sys}}$ is an infinite dimensional closed subspace of $\mathcal{T}_{\text{C}}$. Let $f \in D_{\text{sig}}$ and $T_2 \in D_{\text{sys2}}$ be arbitrary but fixed. Further, let $\hat{h}_T = U^{-1}\hat{h}_{T_2}$. For $N \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{k=-N}^{N} f(k) h_{T_2}(t-k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_{T_2}(\omega)\,\mathrm{e}^{i\omega t} \sum_{k=-N}^{N} f(k)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (U^{-1}\hat{h}_{T_2})(\omega) \sum_{k=-N}^{N} f(k)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_T(\omega) \sum_{k=-N}^{N} f(k)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega \\ &= \sum_{k=-N}^{N} f(k) h_T(-k). \end{aligned} \tag{8}$$

Since $T \in D_{\text{sys}}$, it follows from (8) and (7) that

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_{T_2}(t-k) \right| = \infty. \qquad \square$$

Proof of Corollary 2. The proof of Corollary 2 is similar to the proof of Corollary 1.

From Theorem 1 we know that there exist an infinite dimensional closed subspace $D_{\text{sig}} \subset \mathcal{PW}_\pi^1$ and an infinite dimensional closed subspace $D_{\text{sys}} \subset \mathcal{T}_{\text{C}}$, such that for all $f \in D_{\text{sig}}$, $f \not\equiv 0$, and all $T \in D_{\text{sys}}$, $T \not\equiv 0$, we have (7). Let $t \in \mathbb{R}$ be arbitrary but fixed, and consider the operator $V\colon L^1[-\pi,\pi] \to L^1[-\pi,\pi], \hat{f} \mapsto \hat{f}\,\mathrm{e}^{-i\,\cdot\,t}$. V is a bounded, linear, and invertible operator with bounded inverse. Let $\mathcal{F}$ denote the Fourier transform operator. Then $D_{\text{sig2}} = \mathcal{F}^{-1}V\mathcal{F}D_{\text{sig}}$ is an infinite dimensional closed subspace of $\mathcal{PW}_\pi^1$. Let $f_2 \in D_{\text{sig2}}$ and $T \in D_{\text{sys}}$ be arbitrary but fixed. Further, let $\hat{f} = V^{-1}\hat{f}_2$. For $N \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{k=-N}^{N} f_2(t-k) h_T(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}_2(\omega)\,\mathrm{e}^{i\omega t} \sum_{k=-N}^{N} h_T(k)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (V^{-1}\hat{f}_2)(\omega) \sum_{k=-N}^{N} h_T(k)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega) \sum_{k=-N}^{N} h_T(k)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega \\ &= \sum_{k=-N}^{N} f(k) h_T(-k). \end{aligned} \tag{9}$$

Since $f \in D_{\text{sig}}$, it follows from (9) and (7) that

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f_2(t-k) h_T(k) \right| = \infty,$$

which completes the proof. $\square$

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