ON THE IMPACT OF SIGNALS TIME-FREQUENCY SPARSITY ON THE LOCALIZATION PERFORMANCE

A. Boudjellal[1,2] *V. D. Nguyen*[1] *K. Abed-Meraim*[1] *A. Belouchrani*[2] *Ph. Ravier*[1]

[1] Univ. of Orléans, PRISME Laboratory, France; [2] Ecole Nationale Polytechnique, Algiers, Algeria.
[1] firstname.lastname@univ-orleans.fr, [2] firstname.lastname@enp.edu.dz

ABSTRACT

In this paper, we investigate the localization performance of far field sources that have sparse time-frequency (T-F) representations. The Cramér-Rao Bound (CRB) under the sparsity assumption is developed and the impact of the T-F sparsity prior on the localization performance is analyzed. In particular, one studies how the different T-F sparsity properties i.e. local SNR level, source supports spreading and source overlapping and orthogonality affect the CRB of the Direction-of-Arrival (DoA) estimation. The obtained results show that the sources T-F orthogonality has the most significant impact on the localization performance. Simulation results are provided to illustrate the concluding remarks made out of this study.

Index Terms— CRB, Time-Frequency, Sparsity, DoA.

1. INTRODUCTION

Blind Source Separation (BSS) and DoA estimation are two problems for which a solution can be derived under a variety of priors such as statistical independence, orthogonality, stationarity, and sparsity. The localization performance depends strongly on the considered prior. In this work, we focus our study on sources with sparse Time-Frequency (T-F) representations. Indeed, exploiting such T-F representations leads to improved source separation and DoA estimation performance [1–7]. Moreover, it has been shown that the T-F based approaches are more efficient than classical methods when applied to BSS or DoA estimation in hard scenarios (e.g., convolutive mixtures [8], underdetermined systems [9–13], dependent sources [14–16], and non-stationary signals [11, 17, 18]). This efficiency is due to the fact that in the T-F domain it is possible to exploit the signals sparse T-F signatures and (eventually their partial) orthogonality [8, 19, 20]. Recent works [21–23] have demonstrated that by assuming that the sources can be sparsely represented in a given domain, source separation/localization can be achieved by exploiting this property. The main benefit of such sparse representation is twofold: first, in overdetermined cases, it significantly improves the estimation of DoA and the localization quality of the sources; second, in underdetermined cases, it transforms the ill-posed separation problem into a resolvable one.

However, to the best of our knowledge, there has been no dedicated analysis on the impact of those priors on the localization performance. In this paper, we propose a thorough analysis of the latter in different scenarios that highlight the situations leading to most significant gains. The effect of the sparsity and the T-F orthogonality priors on the DoA estimation accuracy is investigated through the computation and the analysis of the CRB under the sparsity assumption.

2. PROBLEM FORMULATION

We consider a Uniform Linear Array (ULA) receiving m signals $\mathbf{x}(t)$ from n narrowband, far field, closely spaced sources $\mathbf{s}(t)$, located at DoAs $\alpha_i,\ \ i = 1, \cdots, n$, respectively.

$$\mathbf{x}(t) = \mathbf{A}\ \mathbf{s}(t) + \mathbf{e}(t), \qquad t = 1, 2, ..., N \tag{1}$$

where the noise $\mathbf{e}(t)$, is assumed to be zero mean, circular, Gaussian distributed random vector with covariance matrix $\mathbf{R}_e = \sigma^2 \mathbf{I}_m$ and $\mathbf{A} = [\mathbf{a}_1(\alpha_1), \cdots, \mathbf{a}_n(\alpha_n)]$ is the steering matrix. The source snapshots $\mathbf{s}(t)$ are assumed to be deterministic but unknown. By stacking the N data samples into a single vector, we can write:

$$\mathbf{X} = (\mathbf{I}_N \otimes \mathbf{A})\ \mathbf{S} + \mathbf{E} \tag{2}$$

with $\mathbf{X} = [\mathbf{x}(1)^T, \cdots, \mathbf{x}(N)^T]^T$, $\mathbf{E} = [\mathbf{e}(1)^T, \cdots, \mathbf{e}(N)^T]^T$, $\mathbf{S} = [\mathbf{s}(1)^T, \cdots, \mathbf{s}(N)^T]^T$, and the operator $\otimes$ stands for the Kronecker product. Under the aforementioned assumptions, the observed data is a circular, Gaussian random vector with mean $\mu = (\mathbf{I} \otimes \mathbf{A})\ \mathbf{S}$ and covariance matrix $\mathbf{R}_\mathbf{x} = \sigma^2 \mathbf{I}_{mN}$.

The unknown parameter vector to be estimated is $\mathbf{\Theta} = [\boldsymbol{\theta}^T, \mathbf{\Psi}^T]^T$ where $\boldsymbol{\theta} = [\alpha_1, \cdots, \alpha_n]^T$, and $\mathbf{\Psi}^T$ is a vector of all the nuisance parameters, i.e. $\mathbf{\Psi} = [\Re(\mathbf{S}), \Im(\mathbf{S}), \sigma^2]^T$ where $\Re$ and $\Im$ stand for the real and imaginary parts.

Hence, the impact of the previously mentioned source properties on the DoA estimation accuracy (i.e., the CRB) can be quantified through the computation of the top left $n \times n$ $\boldsymbol{\theta}$-block of the Fisher Information Matrix (FIM) inverse. Under the data model assumptions in (2), the $\boldsymbol{\theta}$-block of the CRB matrix w.r.t. the vector $\mathbf{\Theta}$ is given by [24]:

$$C_{det}(\boldsymbol{\theta}) = \frac{\sigma^2}{2N} \left\{ \Re \left((\mathbf{D}^H \Pi_\mathbf{A}^\perp \mathbf{D}) \odot \mathbf{R}_s \right) \right\}^{-1} \tag{3}$$

where $\mathbf{D} = \frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}^T}$ is the derivative of $\mathbf{A}$ with respect to $\boldsymbol{\theta}$, $\Pi_{\mathbf{A}}^{\perp}$ is the complement of the orthogonal projection onto the range space of $\mathbf{A}$, $\mathbf{R}_s = \frac{1}{N}\sum_{t=1}^{N} \mathbf{s}(t)\mathbf{s}^H(t)$, and the operator $\odot$ stands for the Hadamard product.

3. SPARSE TIME-FREQUENCY REPRESENTATION

Different T-F signal representations exist in the literature [19] including the simplest one given by the Short Time Fourier Transform (STFT). For a non-stationary signal $u(t)$, its STFT $U(k, j)$ is obtained by applying the discrete Fourier transform (DFT) to the T segments[1] of a signal $u(t)$ each of them of length N [16]:

$$\mathbf{U}_j = \mathbf{F}\,\mathbf{u}_j \qquad \mathbf{u}_j = \mathbf{F}^H\,\mathbf{U}_j, \quad j = 1:T \tag{4}$$

where $\mathbf{F}$ denotes the unitary $N \times N$ DFT matrix and $\mathbf{u}_j = [u(N(j-1)+1), \cdots, u(jN)]^T$.

Assuming that the time-frequency representation of the signal $u(t)$ is s-sparse ($u(t)$ is localized in the time-frequency domain), each signal segment can be represented by $s < N$ parameters:

$$\mathbf{u}_j = \mathbf{F}^H\,\mathbf{U}_j = \mathbf{F}^s\,\mathbf{c}_j, \quad j = 1:T \tag{5}$$

x where vector $\mathbf{c}_j$ contains only the s non-zero entries of the N-element vector $\mathbf{U}_j$ and the matrix $\mathbf{F}^s$ is composed of the corresponding s columns of $\mathbf{F}^H$. It follows that the NT samples of the signal $u(j)$ can be represented using only sT time-frequency points:

$$\mathbf{u} = \mathbf{H}\,\mathbf{c} \tag{6}$$

where $\mathbf{u} = [u(1), \cdots, u(NT)]^T$, $\mathbf{c} = \left[\mathbf{c}_1^T, \cdots, \mathbf{c}_T^T\right]^T$, are the original and T-F activation coefficient, respectively, and $\mathbf{H} = \mathbf{I} \otimes \mathbf{F}^s$ is the dictionary matrix.

Now, let us consider n sources $S_i(k, j), i = 1, \cdots, n$, all localized in the time-frequency domain:

$$\mathbf{s}_i = \mathbf{H}_i\,\mathbf{c}_i, \ \ i = 1, \cdots, n \tag{7}$$

Based on (6) and (7), the time-domain representation of the vector $\mathbf{S} = \left[\mathbf{s}(1)^T, \cdots, \mathbf{s}(N)^T\right]^T$ is given by:

$$\mathbf{S} \;\; = \;\; \overline{\mathbf{H}}\,\mathbf{c} \tag{8}$$

with a nsT-dim sparse[2] coefficient vector $\mathbf{c} = \left[\mathbf{c}_1^T, \cdots, \mathbf{c}_n^T\right]^T$ and a $nNT \times nsT$ dictionary matrix

$$\overline{\mathbf{H}} = [\overline{\mathbf{H}}_1, \cdots, \overline{\mathbf{H}}_n] = \left[\begin{array}{c|c|c} \mathbf{H}_1(1, 1:sT) & & \mathbf{0} \\ \mathbf{0} & & \mathbf{H}_n(1, 1:sT) \\ \vdots & \cdots & \vdots \\ \mathbf{H}_1(NT, 1:sT) & & \mathbf{0} \\ \mathbf{0} & & \mathbf{H}_n(NT, 1:sT) \end{array}\right]$$

[1]For simplicity sake, we suppose that the T segments of $u(t)$ used to compute its STFT are non-overlapping.

[2]For notational simplicity & w.l.o.g., we assumed all sources have the same sparsity index $'s'$.

Replacing (8) in (2), we get the following sparse model:

$$\mathbf{X} = (\mathbf{I}_N \otimes \mathbf{A})\,\overline{\mathbf{H}}\,\mathbf{c} + \mathbf{E} \tag{9}$$

4. IMPACT OF THE SPARSE T-F REPRESENTATION

4.1. T-F sparsity based CRB

The aim is to analyze the impact of the sparsity on the DoA estimation performance considering the sparse representations of the original sources. In the following, we develop the CRB under the sparse representation model in (9).

Theorem 1 *Under the sparsity assumption, the $\boldsymbol{\theta}$-block of the CRB, w.r.t. the extended parameter vector $\Theta^T = [\boldsymbol{\theta}, \Re(\mathbf{c}), \Im(\mathbf{c}), \sigma^2]$ (assuming $\mathbf{H}$ known) is given by:*

$$C_{sparse}(\boldsymbol{\theta}) = \frac{\sigma^2}{2}\left\{\Re\left((\overline{\mathbf{S}}^H\overline{\mathbf{S}}) \odot (\mathbf{D}^H\mathbf{D}) - (\overline{\mathbf{S}}^H\mathbf{H}) \diamond (\mathbf{D}^H\mathbf{A}) \left((\mathbf{H}^H\mathbf{H}) \diamond (\mathbf{A}^H\mathbf{A})\right)^{-1} (\mathbf{H}^H\overline{\mathbf{S}}) \diamond (\mathbf{A}^H\mathbf{D})\right)\right\}^{-1} \tag{10}$$

with $\overline{\mathbf{S}} = [\mathbf{s}(1), \cdots, \mathbf{s}(N)]^T$, and the operator $\diamond$ stands for the Khatri-Rao product.

Based on this general result, we focus now on particular but insightful situations given by Theorem 2:

Theorem 2 *Knowing $\mathbf{H}$, the sparsity property has no impact on the DoA estimation accuracy for the two following cases:*

- *The mono source case:* $\quad C_{sparse}^{1d}(\alpha) = C_{det}^{1d}(\alpha)$
- *The full overlapping signals case: If the n source signals share exactly the same support ($\mathbf{H}_i = \mathbf{H}, \ \ i = 1, \cdots, n$) then $C_{sparse}(\boldsymbol{\theta}) = C_{det}(\boldsymbol{\theta})$*

Nevertheless, if the supports of the n sources are completely disjoint (i.e. $\mathbf{H}_i^H\mathbf{H}_j = 0$ for all $1 \leq i \neq j \leq n$) then the source signals can be localized with the mono source performance i.e.

$$C_{sparse}(\alpha_i) = C_{det}^{1d}(\alpha) = \frac{1}{2N\mathit{SNR}\,(\mathbf{d}^H\Pi_{\mathbf{a}}^{\perp}\mathbf{d})}$$

where $SNR = \mathbf{S}^H\mathbf{S}/(N\sigma^2)$.

Proof 1 *See [25]. Due to the space limitation, the derivations are omitted in this paper.*

4.2. Impact of the Sparse T-F Representation

The T-F sparsity of the sources induce different properties that are discussed briefly in this section before their investigation through CRB analysis in section 5.

- Contextual[3] sparsity: It has been stated in [26, 27] that the contextual sparsity priors have no effect on the localization performance in the deterministic case. It follows that if the source supports in the T-F domain are not known, it will not be possible to take advantage from the sparse T-F representation.

- Sparsity as information: The sparse T-F signatures result in partial (or possibly total) orthogonality of the sources which enables source discrimination and leads to high localization performance if this a priori information is considered in the data model. However, for the mono source case, there is no need for such 'source discrimination' and hence the sparsity has no impact on the localization performance[4] for the ULA case considered herein, the CRB expression in (3) coincides with the one in (10).

- Overlapping impact: it has been shown that the underdetermined source separation/localization is made possible based on the non-overlapping property. If the source signals are completely disjoint in T-F domain, they can be individually localized reaching the mono source performance. When the source supports are completely overlapped, the orthogonality assumption is no longer valid and hence no performance gain will be obtained through the sparsity prior.

  In intermediate situations to quantify the impact of the partial source overlapping on the localization performance, two aspects need to be considered: the overlapping size ratio (the relative size of the overlapping area to the total size of the source supports in the T-F domain) and the overlapping power ratio (the power of the overlapping part of each source to its total power). For a highly overlapping case (the size of the shared support is much larger than the non-overlapping supports or the power of the overlapping part is much higher than the power of the non-overlapping parts). In that context, the gap between the sparse case and the non sparse case decreases significantly.

5. RESULTS AND ANALYSIS

The contributions of the sparse T-F representation prior on the sources localization performance is illustrated in this section through some simulation experiments. We consider two narrowband source signals with DoAs α and $\alpha + \delta$, respectively, received by an ULA of $m = 5$ antennas. Two uniform i.i.d sources are generated in the T-F domain on a rectangular support (see figure 1) corresponding to $N_f = 8$ frequency bins by $N_t = 8$ time point (64 time-frequency points for each source). The frequency overlapping between the two source supports can vary from 0 (different supports) to 8 frequency bins (the same support for the two sources). Unless otherwise specified, the number of points to compute the Inverse Fourier Transform N is equal to 32, the number of snapshots is $T = 32$, and the signal-to-noise-ratio (SNR) is set equal to $20\ dB$. The source power is normalized to unity (i.e. $\mathbf{s}_i^H \mathbf{s}_i / N = 1$) and the SNR is controlled through the variation of the noise power.

In the first simulation scenario, one source case is considered. The localization performance is evaluated with and without sparsity prior. As illustrated in figure 2, the sparsity prior has no impact on the localization performance for the mono source case. Indeed, one can show that both CRB expressions (for sparse and non sparse cases) coincide.

With contrast to the mono source case, the T-F sparsity has a strong impact on the localization performance when two non-overlapping sources are considered. In this case the performance are similar to those of the mono source case (since the sources have different supports and hence can be separated by T-F masking). The performance of the sparse case decreases when the overlapping zone increases until it reaches the performance of the non-sparse case when the two sources have the same support (completely overlapped sources). We can also see that for only one non-overlapping frequency bin, one gains approximately $30dB$ in localization performance.

Another aspect that has to be considered for the evaluation of the overlapping impact on the localization performance is the relative power of the overlapping part of each source w.r.t. the power of its non-overlapping part. In this second simulation scenario, the overlapping length is set to 4 frequency bins (a 50% overlap) and the signal-to-overlapping-ratio[5] (SOR) of the first source is varied from $-60\ dB$ to $40dB$ and the SOR of the second source is set equal to 1. The behavior of localization performance for the sparse and non-sparse cases are depicted in figure 3. The performance is similar to the mono-source case when the SOR is higher than $0dB$ and is close to those of the two sources, non-spare case when the SOR is low ($-40dB$). Indeed, this situation represents the cases, considered for example for the underdetermined audio source separation, where at each T-F point only one source signal has a 'non negligible' energy contribution (i.e. high SOR).

The sparsity degree defined as the ration s/N is an important factor for the quantification of the T-F sparsity impact on the localization performance. Figure 4 shows its effect on the CRB. One can see that with a sparsity degree of 50% (or less) the performance of the mono source case is almost reached.

From the three simulation scenarios, it can be concluded

[3] By contextual prior, we mean that it is considered only for the source generation process and not considered as known assumption for the performance derivation.

[4] This is a non obvious result as many authors claimed that the local SNR increase, due to the spreading of noise in the T-F domain, leads to localization performance improvement.

[5] The SOR is equal to the ratio of the power of the non-overlapping part over the power of the overlapping part.

that the prior given by the sparse representation has a significant positive impact on the localization performance when two sources have small non-overlapping areas with favorable SOR and low sparsity degree.

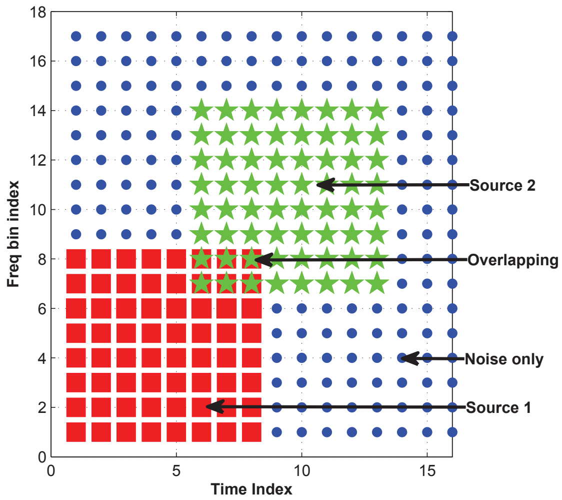

Fig. 1. Source generation scenario in the T-F domain.

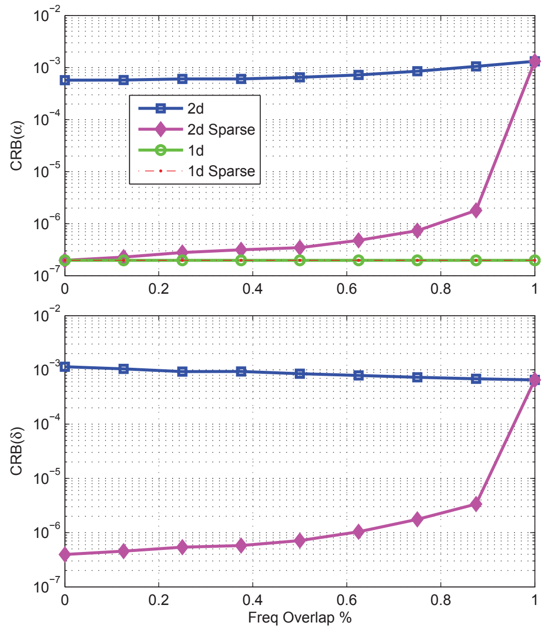

Fig. 2. Impact of the sparsity and overlapping on performance: 1d (resp. 2d) refers to the mono-source (resp. 2 sources) case.

6. CONCLUSION

Many authors have observed that the separation or localization performance of non stationary sources can be significantly improved thanks to their sparsity in the T-F domain. In this work, we investigated this issue through the derivation of the CRB expression for the DoA estimation. Based on this analysis the following remarks can be drawn:

- The T-F sparsity results in local SNR improvement (since the noise power is spread over the whole T-F domain while the signal power is localized in a small area). This local SNR improvement leads to performance gain for some existing localization methods (e.g. MUSIC method [5]). However, for the performance bounds, this local SNR improvement has little if no impact at all (one can see it from the mono-source CRB where the sparse and non-sparse cases coincide).
- A significant gain due to the sources T-F sparsity is obtained when the latter have known 'non-overlapping' regions in the T-F domain. Even when the non-overlapping regions represent a very small part of the T-F domain (10% or less), a strong gain is observed and the localization performance is close to the one we get in the mono source case.
- If the sources are sparse but with unknown T-F supports (i.e. contextual sparsity), the mentioned localization gain is lost. Practically, this means that we should 'detect' the regions in the T-F domain where the sources are active before using such information for the localization or source separation. Note that this approach is already considered by many localization or separation methods developed for the underdetermined case.

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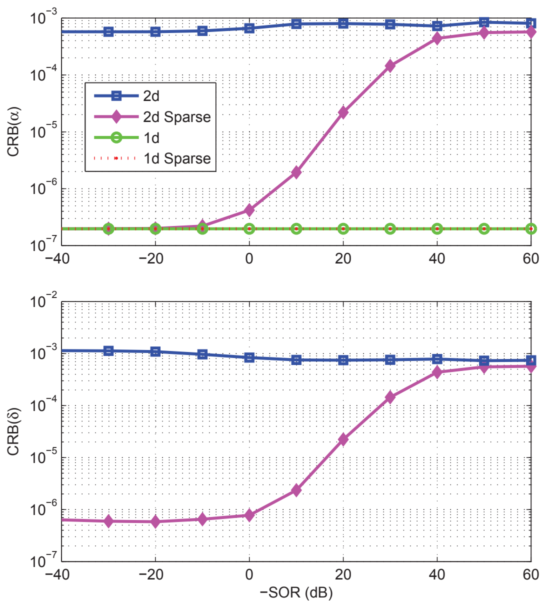

Fig. 3. Impact of the sparsity and overlapping on performance.

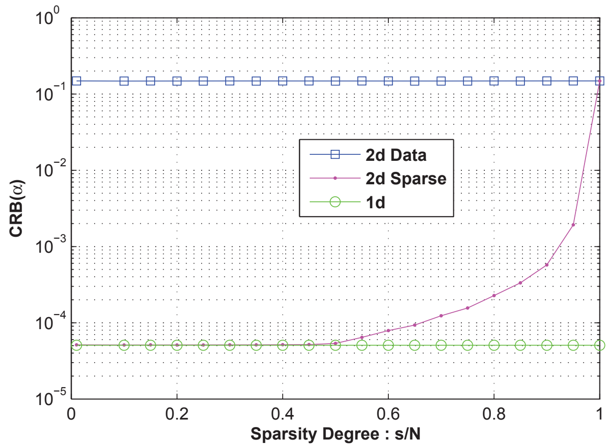

Fig. 4. Impact of the sparsity degree on CRB(α) with 50% of overlap.

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