

# Design of Binary Sequences with Low PSL/ISL

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**Abstract**—In this paper the long standing major challenge of designing binary sequences with good (aperiodic) autocorrelation properties in terms of Peak Sidelobe Level (PSL) and Integrated Sidelobe Level (ISL) is considered. The problem is formulated as a bi-objective Pareto optimization forcing the binary constraint at the design stage. An iterative novel FFT-based approach exploiting the coordinate descent method is devised to deal with the resulting optimization problem which is non-convex and NP-hard in general. Simulation results illustrate that the proposed algorithm can outperform some counterparts providing sequences with desirable PSL as well as ISL.

**Index Terms**—Radar, Waveform Design, Peak Sidelobe Level (PSL), Integrated Sidelobe Level (ISL), Binary Phase Codes.

## I. INTRODUCTION

Binary sequences with low Peak Sidelobe Level (PSL) and Integrated Sidelobe Level (ISL) in aperiodic autocorrelation function have wide applications in active sensing and communication systems [1], [2] being their implementation quite simple. In radar range compression, low PSL binary codes are employed to avoid masking of weak targets in range sidelobes of a strong return [3], [4]. Besides, sequences with low Integrated Sidelobe Level (ISL) are exploited to mitigate the deleterious effects of distributed clutter returns close to the target of interest [5].

Some well-known binary sequences are the Barker codes,  $M$ -sequences, Gold codes, Kasami codes, and Minimum Peak Sidelobe (MPS) sequences. The Barker sequences shares excellent autocorrelation properties but, unfortunately, are limited to length 13.  $M$ -sequences, are renowned for their ideal periodic autocorrelation function but have no constraints/guarantees on the sidelobes of their aperiodic autocorrelation function; hence, they are almost impractical in radar applications (similarly Gold codes and Kasami sequences) [1]. Unlike the case of the periodic correlation, it is not possible to construct sequences with an exact impulsive aperiodic autocorrelation. Therefore, a brute-force approach to obtain good sequences is to perform an exhaustive search, viable especially when the alphabet size is small, i.e., binary case. In this respect, MPS sequences are the best binary codes in terms of PSL (known up to length 105) which are obtained via global search through some supercomputers; a summary of the best known binary sequences is presented in [6]. When the code length increases, it becomes almost impossible to perform the global search. In these situations, derivation of analytical methods to design optimal or nearly-optimal sequences are valuable. To this end, CAN algorithm [7], which minimizes an objective almost equivalent to the ISL, provides high quality continuous phase sequences. However, when phase quantization is implemented, the performance of CAN gets worse especially when the

constellation size is small. In the case of discrete phase code design, three important algorithms attempting to minimize the ISL have been proposed in [8]–[10]. The CANARY [8] (which is an extension of the CAN algorithm) proposes an adhoc method for designing complementary sets of sequences. The Iterative Twisted appROXimation (ITROX) [9] method can be used to obtain sequences (or complementary sets of sequences) possessing good periodic (ITROX-P) or aperiodic (ITROX-AP) correlation properties with continuous/discrete phases. The MBIQ&L algorithm [10] provides an effective approach to design continuous/discrete phase-only modulated waveforms sharing a desired range-Doppler response. Notice that the The MBIQ&L and ITROX are highly-computational demanding and useful to build code libraries.

In this paper, we propose an attractive iterative method to design binary sequences jointly optimizing PSL and ISL according to a Pareto multi-objective optimization framework<sup>1</sup>. It is worth observing that, the synthesis of optimized binary codes in terms of PSL fills a relevant gap in the open literature and this is indeed the main technical contribution of this study<sup>2</sup>. The problem is formulated as a bi-objective optimization where a binary constraint is forced at the design stage. To tackle the resulting non-convex and, in general NP-hard problem, an iterative procedure based on the Coordinate Descent (CD) method is introduced. In each iteration of the proposed algorithm, the solution of a non-convex min-max problem is handled via a DFT-based procedure. Additionally, the selected weighted sum of the ISL and PSL based metrics decreases with iterations until convergence.

The rest of this paper is organized as follows. Section II presents the problem formulation. In Section III, the CD-based method is devised together with a technique aimed at solving the optimization problem involved in each iteration. Section IV is devoted to numerical examples. Finally, concluding remarks are given in Section V.

### A. Notation

We adopt the notation of using bold lowercase letters for vectors and bold uppercase letters for matrices. The transpose, the conjugate, and the conjugate transpose operators are denoted by the symbols  $(\cdot)^T$ ,  $(\cdot)^*$ , and  $(\cdot)^H$  respectively. The  $l_p$ -norm of a vector  $\mathbf{x}$  is denoted by  $\|\mathbf{x}\|_p$ . The letter  $j$  represents the imaginary unit (i.e.,  $j = \sqrt{-1}$ ), while the letter  $i$  often

<sup>1</sup>The PSL and the ISL are the two most important measures quantifying the quality of the autocorrelation function.

<sup>2</sup>Notice that there exist algorithms in the open literature for continuous/discrete phase ISL minimization but they usually do not perform well at the binary case [11].

serves as index. For any  $x \in \mathbb{R}$ ,  $|x|$  and  $\arg(x)$  represent the modulus and the argument of  $x$ , respectively. The abbreviation “s.t.” stands for “subject to”.

## II. PROBLEM FORMULATION

Let  $\mathbf{x} = [x_1, x_2, \dots, x_N]^T \in \mathcal{R}^N$  be the transmitted fast-time radar binary code vector with  $N$  being the number of coded sub-pulses (code length). The autocorrelation function associated with  $\mathbf{x}$  is defined as

$$r_k = \sum_{i=1}^{N-k} x_i^* x_{i+k}, \quad k = 0, \dots, N-1, \quad (1)$$

and represents the output of the matched filter to  $\mathbf{x}$  when  $\mathbf{x}$  is the input signal. The PSL and ISL metrics, are commonly used to design waveforms with “good” autocorrelation properties [1] and are formally defined as  $\text{PSL} = \max\{|r_k|\}_{k=1}^{N-1}$  and  $\text{ISL} = \sum_{k=1}^{N-1} |r_k|^2$ , respectively. This paper is focused on the design of binary sequences considering simultaneously the PSL and the ISL as performance indices. From an analytical point of view the problem can be formulated as the following constrained bi-objective optimization,

$$P \begin{cases} \min_{\mathbf{x}} & f_1(\mathbf{x}), f_2(\mathbf{x}) \\ \text{s.t.} & x_i \in \{-1, 1\}, \quad i = 1, \dots, N \end{cases} \quad (2)$$

where  $f_1(\mathbf{x}) = \max\{|r_k|\}_{k=1}^{N-1}$  and  $f_2(\mathbf{x}) = \sum_{k=1}^{N-1} |r_k|^2$ . In a multi-objective optimization framework, usually a feasible solution that minimizes all the objective functions simultaneously does not exist [12]. Accordingly, the goal is to find the *Pareto-optimal* solutions to (2) that is in general a formidable task. A viable means to obtain the above solutions is the scalarization technique which exploits as objective a specific weighted sum between  $f_1(\mathbf{x})$  and  $f_2(\mathbf{x})$ . Specifically, defining the function  $f_\theta(\mathbf{x})$ , parameterized in the Pareto weight  $\theta \in [0, 1]$ ,

$$\begin{aligned} f_\theta(\mathbf{x}) &= \theta f_1(\mathbf{x}) + (1 - \theta) f_2(\mathbf{x}) \\ &= \max_k \left[ \theta |r_k| + (1 - \theta) \sum_{l=1}^{N-1} |r_l|^2 \right] \end{aligned} \quad (3)$$

scalarization leads to the design problem

$$P^\theta \begin{cases} \min_{\mathbf{x}} & f_\theta(\mathbf{x}) \\ \text{s.t.} & x_i \in \{-1, 1\}, \quad i = 1, \dots, N \end{cases} \quad (4)$$

Remarkably,  $P^\theta$  reduces to pure ISL (PSL) minimization setting  $\theta = 0$  ( $\theta = 1$ ). Moreover, for any  $\theta$ , an optimal solution to (4) is a Pareto-optimal point to Problem (2) (see [13]–[15] and references therein for details).

## III. CD-BASED RADAR CODE OPTIMIZATION

This section introduces an iterative algorithm based on the CD minimization procedure [16] (also known as alternate optimization [17]) to sequentially optimize the objective over one variable keeping fixed the others. Otherwise stated, according to the CD approach, the minimization of a multivariable function is pursued optimizing it along one direction at a time

[16], [18]. With reference to (4), at each iteration, a code entry is selected as variable to optimize leading to the following problems at step  $n + 1$

$$P_{d, \mathbf{x}^{(n)}}^\theta \begin{cases} \min_{x_d} & f_\theta(x_d; \mathbf{x}_{-d}^{(n)}) \\ \text{s.t.} & x_d \in \{-1, 1\} \end{cases} \quad (5)$$

where  $x_d$  is the variable to be optimized,  $\mathbf{x}_{-d}^{(n)} = [x_1^{(n)}, \dots, x_{d-1}^{(n)}, x_{d+1}^{(n)}, \dots, x_N^{(n)}]^T \in \mathcal{R}^{N-1}$ , indicates the remaining code entries, and

$$f_\theta(x_d; \mathbf{x}_{-d}^{(n)}) = f_\theta(x_1^{(n)}, \dots, x_{d-1}^{(n)}, x_d, x_{d+1}^{(n)}, \dots, x_N^{(n)}). \quad (6)$$

Thus, denoting by  $x_{d, n+1}^*$  the optimal solution to  $P_{d, \mathbf{x}^{(n)}}^\theta$ , the optimized radar code at step  $n + 1$  is  $\mathbf{x}^{(n+1)} = [x_1^{(n)}, \dots, x_{d-1}^{(n)}, x_{d, n+1}^*, x_{d+1}^{(n)}, \dots, x_N^{(n)}]^T$ . As a result, starting from an initial code  $\mathbf{x}^{(0)}$ , a sequence  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \dots$  of radar codes are obtained iteratively. A summary of the proposed approach can be found in **Algorithm 1**. Notice that, the monotonic property of the CD technique along with the fact that the objective function is bounded (from below) are sufficient to prove the convergence of the sequence of objective values. It is also worth pointing out that the Maximum Block Improvement (MBI) updating rule<sup>3</sup> [19] can be used in place of the cyclic one (actually involved in **Algorithm 1**) to ensure the convergence of the algorithm to a stationary point. In practice, a final optimized code can be obtained refining the solution provided **Algorithm 1** through the MBI-modification. To proceed further, let us make explicit the functional depen-

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### Algorithm 1 Binary Code Design (BCD) with Low PSL/ISL

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**Input:** Initial code  $\mathbf{x}_0 = [x_1^{(0)}, x_2^{(0)}, \dots, x_N^{(0)}]^T$ ,  $x_i \in \{-1, 1\}$ ,  $i = 1, \dots, N$ ,  $\theta \in [0, 1]$ , and required improvement  $\epsilon$ ;

**Output:** Optimal solution  $\mathbf{x}^*$ ;

1) **Initialization.**

- Compute the initial objective value  $f_\theta(x_1^{(0)}, x_2^{(0)}, \dots, x_N^{(0)})$  using equation (3);
- Set  $d := 1$  and  $n := 0$ ;

2) **Improvement.**

- Solve  $P_{d, \mathbf{x}^{(n)}}^\theta$  obtaining  $x_d^*$ ;
- Set  $n := n + 1$  and  $\mathbf{x}^{(n)} = [x_1^{(n)}, \dots, x_{d-1}^{(n)}, x_d^*, x_{d+1}^{(n)}, \dots, x_N^{(n)}]^T$ ;

3) **Stopping Criterion.**

- If  $|f_\theta(\mathbf{x}^{(n)}) - f_\theta(\mathbf{x}^{(n-1)})| < \epsilon$ , stop. Otherwise, update  $d$ , i.e., if  $d < N$ ,  $d = d + 1$  else  $d = 1$ , and go to the step 2;

4) **Output.**

- Set  $\mathbf{x}^* = \mathbf{x}^{(n)}$ .
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<sup>3</sup>The MBI method is an iterative algorithm known to achieve excellent performance in the maximization of real polynomial functions subject to spherical constraints [10]. It is proved that any cluster point of the sequence produced by the MBI method is a stationary point for the considered optimization problem [19].

dence of the objective function in  $P_{d,\mathbf{x}^{(n)}}^\theta$ , i. e.,  $f_\theta(x_d; \mathbf{x}_{-d}^{(n)})$ , over the optimization (real binary) variable  $x_d$

$$r_k(x_d) = x_d \left( x_{d+k}^{(n)} \mathbf{1}_A(d+k) + x_{d-k}^{(n)} \mathbf{1}_A(d-k) \right) + \sum_{i=1, i \neq d, d-k}^{N-k} x_i^{(n)} x_{i+k}^{(n)}, \quad (7)$$

where  $\mathbf{1}_A(\cdot)$  denotes the indicator function of the set  $A = \{1, 2, \dots, N\}$ , i.e.,  $\mathbf{1}_A(\alpha) = 1$  if  $\alpha \in A$ , otherwise  $\mathbf{1}_A(\alpha) = 0$ . Defining the real coefficients,  $a_{dk} = x_{d+k}^{(n)} \mathbf{1}_A(d+k) + x_{d-k}^{(n)} \mathbf{1}_A(d-k)$  and  $c_{dk} = \sum_{i=1, i \neq d, d-k}^{N-k} x_i^{(n)} x_{i+k}^{(n)}$ , the auto-correlation function with the explicit  $x_d$ -dependence can be written as

$$r_k(x_d) = a_{dk}x_d + c_{dk}, \quad k = 1, \dots, N-1. \quad (8)$$

Thus, the optimization problem  $P_{d,\mathbf{x}^{(n)}}^\theta$  can be recast as,

$$\begin{cases} \min_{x_d} & \max_k \left[ \theta |a_{dk}x_d + c_{dk}|^2 + (1-\theta) \sum_{l=1}^{N-1} |a_{dl}x_d + c_{dl}|^2 \right] \\ \text{s.t.} & x_d \in \{-1, 1\} \end{cases}$$

which is a non-convex constrained min-max problem. In the next subsection, an efficient algorithm to tackle  $P_{d,\mathbf{x}^{(n)}}^\theta$  is derived. This procedure paves the way for the design of arbitrary discrete phase codes.

#### A. Binary Code Design

In this subsection, an efficient procedure to solve  $P_{d,\mathbf{x}^{(n)}}^\theta$  is developed exploiting Discrete Fourier Transform (DFT). To this end, notice that in terms of  $\phi_d = \arg(x_d) \in \{0, \pi\}$ ,  $P_{d,\mathbf{x}^{(n)}}^\theta$  can be recast as,

$$\tilde{P}_{d,\phi_d}^\theta \begin{cases} \min_{\phi_d} & \max_k g_\theta(\phi_d) \\ \text{s.t.} & \phi_d \in \{0, \pi\} \end{cases} \quad (9)$$

where  $g_\theta(\phi_d) = \theta |\tilde{r}_k(\phi_d)|^2 + (1-\theta) \sum_{l=1}^{N-1} |\tilde{r}_l(\phi_d)|^2$ , and  $|\tilde{r}_k(\phi_d)|^2 \triangleq |a_{dk}e^{j\phi_d} + c_{dk}|^2$ . The following lemma provides a key result to tackle Problem (9).

**Lemma III.1.** Let  $\mathbf{v}_{dk} = [|\tilde{r}_k(\bar{\phi}_1)|^2, |\tilde{r}_k(\bar{\phi}_2)|^2]^T \in \mathcal{R}^2$ , with  $\bar{\phi}_i = \pi(i-1)$ ,  $i = 1, 2$ , and  $\zeta_{dk} = [a_{dk}, c_{dk}]^T \in \mathcal{R}^2$ . Then

$$\mathbf{v}_{dk} = |\text{DFT}(\zeta_{dk})|^2, \quad (10)$$

where  $\text{DFT}(\zeta_{dk})$  is the two points DFT of the vector  $\zeta_{dk}$  and the square modulus is element wise.

Now, defining the matrix  $\mathbf{U} \in \mathbb{R}^{(N-1) \times 2}$  whose  $k$ th row is

$$\mathbf{u}^k = \theta \mathbf{v}_{dk}^T + (1-\theta) \sum_{l=1}^{N-1} \mathbf{v}_{dl}^T \in \mathbb{R}^2, \quad k = 1, \dots, N-1,$$

the optimal solution to  $\tilde{P}_{d,\phi_d}^\theta$  is given by

$$\phi_d^* = \pi(i^* - 1), \quad (11)$$

where  $i^* = \arg \min_{i=1,2} \left\{ \max(\mathbf{u}_i) \right\}$ , and  $\mathbf{u}_i \in \mathbb{R}^{(N-1)}$  is the  $i$ th column of  $\mathbf{U}$ . Hence, based on Lemma III.1 and (11),

the optimal phase code entry can be efficiently computed as  $x_d^* = e^{j\phi_d^*}$  using DFT. The proposed approach is reported in **Algorithm 2**.

**Remark 1.** **Algorithm 2** needs the evaluation of  $N-1$  different two points DFTs. Therefore (computed the parameters) the computational complexity order is  $\mathcal{O}(N)$  [20].

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#### Algorithm 2 Binary Code Entry Optimization

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**Input:** Initial code vector  $\mathbf{x}^{(n)}$ , code entry  $d$  and  $\theta$ ;

**Output:** Optimal solution  $x_d^*$ ;

- 1) Set for all  $k = 1, \dots, N-1$ 
    - $a_{dk} = x_{d+k}^{(n)} \mathbf{1}_A(d+k) + x_{d-k}^{(n)} \mathbf{1}_A(d-k)$  and  $c_{dk} = \sum_{i=1, i \neq d, d-k}^{N-k} x_i^{(n)} x_{i+k}^{(n)}$ ;
    - $\zeta_{dk} = [a_{dk}, c_{dk}]^T$  and  $\mathbf{v}_{dk} = |\text{FFT}(\zeta_{dk})|^2$ ;
  - 2) Calculate  $\mathbf{u}^k = \theta \mathbf{v}_{dk}^T + (1-\theta) \sum_{l=1}^{N-1} \mathbf{v}_{dl}^T \in \mathbb{R}^2$ ,  $k = 1, \dots, N-1$  and  $\boldsymbol{\omega}_d = [\max\{\mathbf{u}_1\}, \max\{\mathbf{u}_2\}]^T$ ;
  - 3) Find the index  $i^*$  where  $\boldsymbol{\omega}_d$  is minimum;
  - 4) Set  $x_d^* = e^{j\phi_d^*}$  with  $\phi_d^* = \pi(i^* - 1)$ .
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#### B. Algorithm Initialization

The solution obtained via the designed method depends evidently on the initial sequence. As a result, the development of a heuristic approach that can be used to provide high quality starting points is valuable. To this end, recall that the minimization of the  $l_p$ -norm of the autocorrelation vector  $[r_1, r_2, \dots, r_{N-1}]$  allows to trade-off ISL and PSL values of the devised sequence as the value of  $p$  increases [21]–[23]. Besides, the PSL coincides with the limit as  $p \rightarrow \infty$  of the autocorrelation vector  $l_p$ -norm. According to the above considerations, a heuristic procedure to obtain binary codes with low autocorrelation  $l_p$ -norm is introduced. In particular, with reference to the PSL metric, a start-stop procedure involving a sequence of  $l_p$ -norm minimization problems with increasing value of  $p$ ,  $p_1 < p_2 < \dots < p_e$  say, is employed similarly to [21]. Specifically, the algorithm is initialized with a binary random sequence and the  $l_p$ -norm minimization starts with  $p = 2$ , i.e.,  $p_1 = 2$ . Then,  $p$  is set to  $p_2$  and the algorithm starts with the solution obtained for  $p = p_1$ , and so on. In general, the  $l_p$ -norm minimization problem for binary codes can be formulated as

$$H^p \begin{cases} \min_{\mathbf{x}} & \sum_{k=1}^{N-1} |r_k|^p \\ \text{s.t.} & x_i \in \{-1, 1\}, \quad i = 1, \dots, N \end{cases} \quad (12)$$

To tackle  $H^p$  the algorithm proposed in [24] is exploited, where each variable block corresponds to one code entry and the surrogate function of [21] is adopted. Specifically, at step  $n+1$  of the iterative procedure, the following optimization problem is considered,

$$H_{d,\mathbf{x}^{(n)}}^p \begin{cases} \min_{x_d} & \sum_{k=1}^{N-1} \tilde{\tau}_k |r_k(x_d)|^2 + \tilde{\lambda}_k |r_k(x_d)| \\ \text{s.t.} & x_d \in \{-1, 1\} \end{cases} \quad (13)$$

where  $\tilde{\tau}_k = \frac{t_n^p - |r_k^{(n)}|^{p-1} |r_k^{(n)}|^{p-1} (t_n - |r_k^{(n)}|)}{(t_n - |r_k^{(n)}|)^2}$ ,  $\tilde{\lambda}_k = \frac{p |r_k^{(n)}|^{p-1} - 2\tilde{\tau}_k |r_k^{(n)}|}{\left(\sum_{k=1}^{N-1} |r_k^{(n)}|^p\right)^{\frac{1}{p}}}$  and  $t_n = \left(\sum_{k=1}^{N-1} |r_k^{(n)}|^p\right)^{\frac{1}{p}}$  with  $[r_1^{(n)}, r_2^{(n)}, \dots, r_{N-1}^{(n)}]^T$  the optimized autocorrelation vector at step  $n$ . In terms of the discrete phase variable  $\phi_d$ ,  $H_{d, \mathbf{x}^{(n)}}^p$  can be cast as

$$H_{d, \phi_d}^p \begin{cases} \min_{x_d} \sum_{k=1}^{N-1} \tilde{\tau}_k |a_{dk} e^{j\phi_d} + c_{dk}|^2 + \tilde{\lambda}_k |a_{dk} e^{j\phi_d} + c_{dk}| \\ s.t. \quad \phi_d \in \{0, \pi\} \end{cases}$$

Hence, using Lemma III.1 and considering the definition of  $\nu_{dk}$  in (10), the optimal  $x_d^*$  can be efficiently obtained as  $x_d^* = e^{j\pi(i^*-1)}$ , with  $i^* = \arg \min_{i=1,2} \{y_i\}$  and  $\mathbf{y} = [y_1, y_2]^T = \sum_{k=1}^{N-1} (\tilde{\tau}_k \nu_{dk} + \tilde{\lambda}_k \sqrt{\nu_{dk}})$ .

#### IV. PERFORMANCE ANALYSIS

This section is devoted to the performance analysis of the proposed algorithm for Binary Code Design (BCD) exploiting PSL and ISL as figure of merits. For comparison purposes the behavior of the sequences devised via ITROX-AP [9] are reported too. The considered procedures are initialized using the same set of 5 random binary starting codes (drawn from a uniform distribution over the set of the feasible sequences). Hence, the best obtained objective value is reported and the resulting sequence picked up. Finally, the stopping criteria  $|\text{obj}(\mathbf{x}^{(n)}) - \text{obj}(\mathbf{x}^{(n-1)})| \leq 10^{-5}$  is used to terminate all the algorithms.

##### A. PSL Minimization

In this subsection, the ability of the proposed algorithms to synthesize low PSL sequences is assessed. To this end, the Pareto weight is fixed to  $\theta = 1$  and the sequence of  $p$ -values for the selection of the initial starting point is  $2, 2^2, 2^3, \dots, 2^{13}$ , i.e.,  $p_i = 2^i$ ,  $i = 1, \dots, 13$ .

In Fig. 1, the PSL versus the code length  $N$  of BCD and ITROX-AP are reported. To highlight the quality of BCD algorithm also the PSL of MPS sequences, obtained via exhaustive search up to the length of 105, is shown in the figure. The plot clearly illustrates the effectiveness of our approach. Indeed, BCD significantly outperforms ITROX-AP. Besides it provides a PSL quite close to the global optimum of MPS sequences but with a much lower computational complexity and without restrictions to the maximum code length. This last feature is particularly appealing since the higher  $N$  the higher the pulse compression. Interestingly, BCD provides in some circumstances the global optimal solution (see in Fig. 1 the points where BCD and MPS coincide).

In Fig. 2 the autocorrelation function devised via BCD and ITROX-AP for sequence length 126 is displayed. Therein, the PSL of BCD is equal to 8 whereas the PSL of ITROX-AP is 12 which further confirms the effectiveness of the new framework.

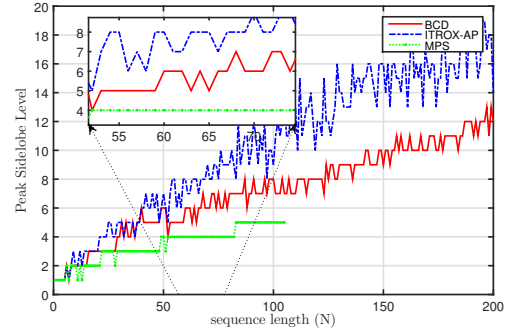


Fig. 1. Comparison between the PSL values of binary sequences obtained through BCD, ITROX-AP, and exhaustive search (MPS). For each length, algorithm (with heuristic initialization and  $\theta = 1$ ) and ITROX-AP have been run 5 times and from the resulting 5 PSL values the best one has been chosen.

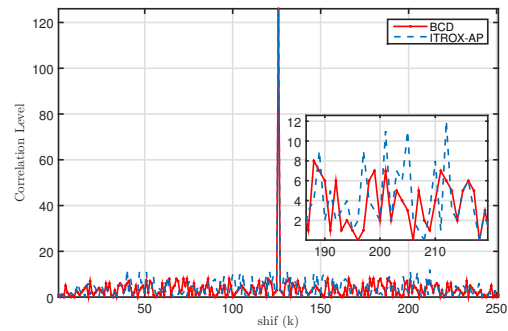


Fig. 2. Autocorrelation function versus delay bin for binary codes of length 126: red curve synthesized via BCD algorithm (with heuristic initialization and  $\theta = 1$ ); blue curve obtained through ITROX-AP.

##### B. ISL Minimization

The performance assessment of BCD for ISL minimization is now considered. In this case,  $\theta = 0$  and the initialization procedure in Subsection III-B is not performed. Fig. 3 shows the ISL versus the code length  $N$  for BCD and ITROX-AP. The result highlights that BCD outperforms ITROX-AP with a maximum ISL gain of 2.25 dB. Remarkably, BCD also provides an ISL close to that of the MPS sequences. In Fig. 4, the autocorrelation function devised via BCD and ITROX-AP for sequence length 126 is reported. In this specific case, BCD provides a ISL-gain over ITROX-AP of 2.1 dB.

##### C. Pareto-Optimal Solution

In this subsection, the impact of the parameter  $\theta$  on the designed code is assessed. Table I reports the PSL and the ISL of the solutions obtained via BCD assuming  $N = 512$  and  $\theta \in \{\theta_1, \dots, \theta_4\}$  with  $\theta_1 = 1$ ,  $\theta_2 = 0.7$ ,  $\theta_3 = 0.3$ , and  $\theta_4 = 0$ . The starting sequence used at  $\theta = \theta_i$  is the code optimized at  $\theta = \theta_{i-1}$ ; also, at  $\theta = \theta_1$  the heuristic approach of Subsection III-B is used. As expected,  $\theta$  trades-off ISL and PSL values. Specifically, the higher  $\theta$  the better the PSL and the worst the ISL, that is a classical feature of bi-objective Pareto curves. Otherwise stated, any solution is a ‘‘Pareto-optimal’’ point.

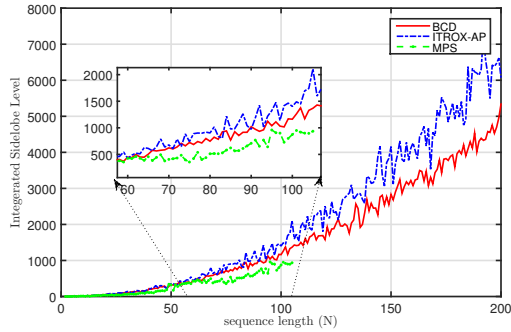


Fig. 3. Comparison among the ISL values of binary sequences obtained through BCD algorithm and ITROX-AP. For each length, BCD algorithm (with  $\theta = 0$ ) and ITROX-AP have been run 5 times and from the resulting 5 ISL values the best one has been chosen.

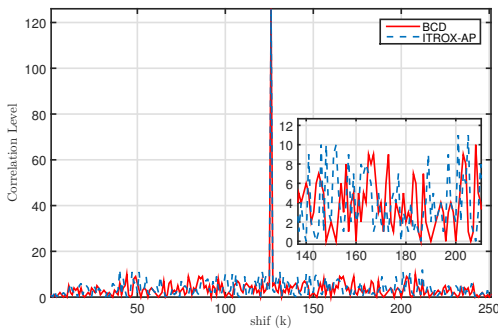


Fig. 4. Autocorrelation function versus delay bin for binary codes of length 126: red curve synthesized via BCD algorithm (with  $\theta = 0$ ); blue curve obtained through ITROX-AP.

V. CONCLUSION

The synthesis of binary codes exhibiting good aperiodic correlation features has been addressed. Specifically, PSL and ISL have been adopted as performance metrics and the design problem has been formulated as a bi-objective optimization. The non-convex and, in general, NP-hard problem resulting from scalarization is handled via a novel iterative procedure based on the CD method and an efficient DFT-based procedure. Finally, a heuristic procedure based  $l_p$ -norm minimization has been introduced to suitably initialize the new convergence-ensured algorithm. Simulation results have illustrated the effectiveness of the new BCD algorithm. Specifically the proposed method can outperform ITROX-AP with reference to both PSL and ISL.

TABLE I  
PSL AND ISL OF "PARETO-OPTIMAL" SOLUTIONS

	$\theta = 0$	$\theta = 0.3$	$\theta = 0.7$	$\theta = 1$
PSL (dB)	13.97	13.80	13.61	13.42
ISL (dB)	45.00	45.04	45.14	46.14

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