Joint Frequency and 2-D DOA Recovery with sub-Nyquist Difference Space-Time Array

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Abstract—In this paper, joint frequency and 2-D direction of arrival (DOA) estimation at sub-Nyquist sampling rates of a multi-band signal (MBS) comprising of \( P \) disjoint narrowband signals is considered. Beginning with a standard uniform rectangular array (URA) consisting of \( M = M_x \times M_y \) sensors, this paper proposes a simpler modification by adding a \( N - 1 \) delay channel network to only one of the sensor. A larger array is then formed by combining the sub-Nyquist sampled outputs of URA and the delay channel network, referred to as the difference space-time (DST) array. Towards estimating the joint frequency and 2-D DOA on this DST array, a new method utilizing the 3-D spatial smoothing for rank enhancement and a subspace algorithm based on ESPRIT is presented. Furthermore, it is shown that an ADC sampling frequency of \( f_s \geq B \) suffices, where \( B \) is the bandwidth of the narrow-band signal. With the proposed approach, it is shown that \( O(MN/4) \) frequencies and their 2-D DOAs can be estimated even when all frequencies alias to the same frequency due to sub-Nyquist sampling. Appropriate simulation results are also presented to corroborate these findings.

Index Terms—Joint frequency-direction of arrival estimation, sub-Nyquist sampling, Space-time array, ESPRIT, Uniform rectangular array, Multiple-delay architecture.

I. INTRODUCTION

Of late, the quest for improving the degrees of freedom with fewer physical elements has drawn considerable attention. Primarily, this is motivated by the fact that over recent years, although the cost of physical elements such as sensors have come down, the deployment (which requires several additional components) and the maintenance cost is still higher. Encouraged by these recent developments, in this paper we consider the important problem of joint frequency and 2-D DOA (i.e., azimuth and elevation) estimation of multi-band signal (MBS) (i.e., multiple disjoint narrow band signals spread within the wide spectrum), under the scenario where the number of sources can exceed the number of sensors respectively. Based on the above methods and the recent advancements in array processing and sub-Nyquist sampling schemes, [10] - [12] proposed methods to address the case of \( M < P \). While [10] employed a nested sensor array [13] based architecture, [11] and [12] considered to use a multi-coset sub-Nyquist sampler [14] at the output of every sensor. In practice, a multi-coset receiver is realized through a multi-channel architecture and hence requires more hardware channels to implement the methods of [11], [12]. Thus, for the problem considered in this paper, the existing methods either requires a newer array geometry or would require more hardware.

In this paper, we assume a standard uniform rectangular array (URA) configuration and propose an efficient method for joint frequency and 2-D DOA estimation for the case of \( M < P \) at sub-Nyquist sampling rates. Based on the idea of [8], the architecture is suitably modified by adding a \( N - 1 \) channel delay network at only one sensor. By assuming the sources to be uncorrelated, we describe a process for obtaining a larger array referred as the Difference Space Time (DST) array. Although URA is a 2-D uniform array, this DST array would be a 3-D uniform array; two dimensions corresponding to spatial delay and the third dimension corresponding to a temporal delay. With this modification, a new rank enhancement method and a corresponding estimation algorithm based on ESPRIT [15] is presented for estimating automatically paired parameters. Later in Section III-D we show that with the proposed approach and for a URA comprising of \( M = M_x \times M_y \) sensors and a single \( N - 1 \) channel delay network (number of ADC channels = \( M + N - 1 \)), upto \( P \leq M(N - 1)/4 \) carrier frequencies and their 2-D DOAs can be determined. Further, it will be shown that if the bandwidth of the narrow-band signal does not exceed \( B \), then a minimum overall sampling rate of \((M + N - 1)B\) would be sufficient to estimate the above mentioned number of carrier frequencies and their 2-D DOAs which shall also be corroborated through simulation results.

II. SIGNAL MODEL AND PROBLEM DESCRIPTION

We assume \( P \) uncorrelated, disjoint, far-field, narrow-band signals which are spread within a wide spectrum of \( F = [0, 1/T] \), impinging on a URA comprising of \( M = M_x \times M_y \) omnidirectional sensors. Let \( x(t) \) denote the combination of \( P \) narrow-band signals referred to as multi-band signal (MBS), which can be expressed as

\[
x(t) = \sum_{p=1}^{P} s_p(t) e^{j2\pi f_p t}
\]  

(1)
The following section describes the proposed approach for these parameters in order to overcome the association problem. Making it difficult to estimate the carrier frequencies from the aliased spectrum. Let us assume a symmetric URA whose sensor array locations are given by

\[ S_{ura} = \{d[m_x, m_y], -[M_x/2, M_x/2, -1, -[M_y/2, M_y/2, -1]\} \]

where \(I\) denotes the identity matrix, \(m_x, m_y \in \mathbb{Z}\) and \(d \leq c/2T\); \(c\) is the wave propagation velocity. Now, the signal observed by the above URA can be expressed in the following form as

\[ x_s(t) = (A_x \otimes A_y) s(t) + \eta(t) \]

where the spatial array manifold matrices for \(k = x\) and \(k = y\),

\[ A_k = [a_k(f_1, \theta_1, \phi_1), \ldots, a_k(f_{P}, \theta_{P}, \phi_{P})] \]

and \(a_k(f_p, \theta_p, \phi_p) = e^{j2\pi d/c(M_x/2)} e^{j2\pi f_pT} \omega_k^x e^{j2\pi \theta_p} \sin \phi_p \omega_k^x \cos \theta_p \sin \phi_p \omega_k^x \cos \theta_p \sin \phi_p \Omega_k \]

\(s(t) = [n_1(t) e^{j2\pi f_1T}, \ldots, n_P(t) e^{j2\pi f_PT}]\).

\(x_s(t)\) denotes the noise vector which is white and uncorrelated with the signal. Let us sample \(x_s(t)\) at a sub-Nyquist sampling rate \(f_s = 1/LT\) i.e., the ADCs samples at every \(t = nLT\), where \(L\) denotes the sub-sampling factor. The sub-sampled signal \(x_s(n)\) can be expressed as

\[ x_s(n) = A_x s(n) + \eta(n). \]

Now, the discrete time Fourier transform (DTFT) of \(x_s(n)\), \(X_s(f)\) can be expressed as \([7]\)

\[ X_s(f) = A_x S(f) + \eta(f) \]

where \(S(f) = [S_1(f), S_2(f), \ldots, S_P(f)]^T\), \(S_p(f)\) denotes the periodic aliased spectrum of the \(p\)th source signal.

As mentioned earlier, \(x(t)\) is assumed to be a wideband signal and hence \(1/T\) will be a large quantity and hence sampling at Nyquist rate may not be feasible. Thus the aim of this paper is to estimate the parameters \(\{f_p, \theta_p, \phi_p\}_{p=1}^P\) at sub-Nyquist sampling under the scenario that the number of sources can exceed the number of sensors i.e., \(P > M\). Further we would like to achieve the above stated goals without enormously increasing the hardware complexity.

Now, observe from \(\{\omega_k^x, \omega_k^y\}_{p=1}^P\), that the carrier frequencies and DOAs appear in non-separable and form and estimating the triple \(\{f_p, \theta_p, \phi_p\}_{p=1}^P\) from these two quantities is not possible. Further, the signal is sampled at sub-Nyquist sampling rates making it difficult to estimate the carrier frequencies from the sub-sampled \(x(n)\). Hence, besides suitably modifying the architecture which can aid in estimating the carrier frequencies, the recovery approach must also facilitate joint estimation of these parameters in order to overcome the association problem. The following section describes the proposed approach for achieving the above stated goals.
B. Difference Space Time Array

Let the combined spatial and temporal delay network DTFT be represented by $X_{st}(f) = [X_s(f), X_t(f)]^T$. $X_{st}(f)$ can be expressed as

$$X_{st}(f) = \begin{bmatrix} A_s & A_t \\ A_s \end{bmatrix} S(f) + N(f). \quad (8)$$

Using the above equation, we form the following covariance matrix

$$R_{xx}^t = \int_{f \in F} X_{st}(f)X_{st}^H(f)df = A_s \left( \int_{f \in F} S(f)S^H(f)df \right) A_s^T + \sigma_n^2 I$$

where $R_{xx}$ denotes the source covariance matrix. Due to the assumption of uncorrelated sources, $R_{xx}$ will be a diagonal matrix with the elements captured by the vector $\lambda = [\sigma_{1x}^2, \sigma_{2x}^2, ..., \sigma_{nx}^2]^T$, where $\sigma_{px}^2$ denotes the power of the $p^{th}$ source. By vectorizing $R_{xx}^t$, it can be expressed as

$$z = \text{vec}(R_{xx}^t) = (A_x^* \otimes A_t) \text{vec}(R_{xx}) + \sigma_n^2 I$$

$$= (A_x^* \otimes A_s)\lambda + \sigma_n^2 I$$

$$= \begin{bmatrix} A_x^* & A_t \\ A_x^* & A_s \\ A_x^* & A_s \\ A_x^* & A_s \end{bmatrix} \lambda + \sigma_n^2 I$$

$$z = \text{vec}(R_{xx}^t) = (A_x^* \otimes A_t) \text{vec}(R_{xx}) + \sigma_n^2 I$$

where ‘$\otimes’$ denotes the conjugate operation, ‘$\otimes’$ and ‘$\odot’$ denotes the Kronecker product and Khatri-Rao product respectively. The simplification from the Kronecker product to Khatri-Rao product is due to the diagonal structure of $R_{xx}$. From the structure of $A_{st}$, it can be observed that $(A_{st}^* \otimes A_{st})$ contains rows corresponding to the difference sensor locations or in other words, $(A_{st}^* \otimes A_{st})$ enumerates an array which we refer as difference array. Furthermore, it can be observed that difference array in general will be larger and would contain several virtual sensors. Let $z_{dst}^{m_x,m_y,m_z} = (m_x, m_y, m_z)$ denotes the sensor locations along the $x, y$ and $z$-axis respectively, denote a subset of $z$ corresponding to the rows of $(A_x^* \otimes A_t)$ and $(A_x^* \otimes A_s)$ which may be written as

$$z_{dst}^{m_x,m_y,m_z} = \begin{bmatrix} A_x^* & A_t \\ A_x^* & A_s \\ A_x^* & A_s \\ A_x^* & A_s \end{bmatrix} \lambda + \sigma_n^2 I.$$

An example of a DST array for $M_x = M_y = N = 3$ is shown in Fig. 2. While the sensors indicated by blue depicts the actual sensors, sensors indicated by red indicates the virtual sensors. The bigger DST array compared to size of the actual physical sensing elements can clearly be noticed from the figure. It is to be noted that although $A_{dst}$ is a bigger array, however, it is a column vector and hence the existing algorithms cannot be directly applied. Thus, in the following section, we present a new approach for estimating parameters with this bigger DST array.

C. Estimation algorithm

In this section, we first outline a 3-D rank enhancing algorithm based on the idea of spatial smoothing [15] and then describe an ESPRIT based algorithm capable of jointly estimating the frequencies and their corresponding 2-D DOAs.

1) Rank enhancing covariance matrix formulation: Let $A_{dst} = A_{x}^* \odot A_{y}^* \odot A_{t}$, where for $k = x$ and $k = y$, $A_k$ is of size $M_k^d \times P$, $M_k^d = [M_k/2]$ and $[A_{dst}]_{m_x,m_y,m_z} = e^{-j2\pi(d/c)(\omega_m-m_x)}$. It is now easy to express $z_{dst}^{m_x,m_y,m_z} = A_{dst}\Delta_x^{m_x} \Delta_y^{m_y} \Delta_z^{m_z}\lambda$, where $\Delta_x^{m_x}, \Delta_y^{m_y}, \Delta_z^{m_z}$ are diagonal matrices whose $(p,p)^{th}$ element is given by $e^{-j2\pi(d/c)x_0 m_x}$, $e^{-j2\pi(d/c)y_0 m_y}$ and $e^{-j2\pi\omega m_z}$ respectively. The rank enhanced covariance matrix of the DST can now be formed as

$$R_{dst} = \begin{bmatrix} [M_x/4]-1 & \vdots & [M_x/4]-1 \\ \vdots & \ddots & \vdots \\ [N/2]-1 & \vdots & [N/2]-1 \end{bmatrix} \sum_{m_x=-[M_x/4]}^{[M_x/4]} \sum_{m_y=-[M_y/4]}^{[M_y/4]} \sum_{m_z=-[N/2]}^{[N/2]} z_{dst}^{m_x,m_y,m_z}(z_{dst}^{m_x,m_y,m_z})^H$$

$$= \begin{bmatrix} [M_x/4]-1 & \vdots & [M_x/4]-1 \\ \vdots & \ddots & \vdots \\ [N/2]-1 & \vdots & [N/2]-1 \end{bmatrix} \sum_{m_x=-[M_x/4]}^{[M_x/4]} \sum_{m_y=-[M_y/4]}^{[M_y/4]} \sum_{m_z=-[N/2]}^{[N/2]} z_{dst}^{m_x,m_y,m_z}(z_{dst}^{m_x,m_y,m_z})^H$$

2) Joint frequency and 2-D DOA estimation: We first determine the singular vectors $U_k$ corresponding to the $P$ largest singular values of $R_{dst}$. In the noise-free setting it can easily be shown that $U_k$ and $A_{dst}$ spans the same subspace and hence $A_{dst} = U_k T_k$, where $T_k$ denotes a full rank transformation matrix of size $P \times P$.

Now, let us define the transformation matrices $\alpha_k^f = [0 \ 1 \ 0 \ -1 \ 0], \alpha_k^t = [1 \ 0 \ -1 \ 0] \in \mathbb{R}^{N-1 \times N}$. Also defined are the transformation matrices $\alpha_k^{\rho}, \alpha_k^{\phi} \in \mathbb{R}^{M_k^d-1 \times M_k^d}$, and $\alpha_k^{\rho}, \alpha_k^{\phi} \in \mathbb{R}^{M_k^d-1 \times M_k^d}$ which are similar to $\alpha_k^f$ and $\alpha_k^t$. Further,
let us define the following

$$\beta_i^f = (I_M \otimes \alpha_f^i) \in \mathbb{R}^{M^2M^2_0(N-1) \times M^2M^2_0} \tag{14}$$

$$\beta_r^f = (I_M \otimes \alpha_r^f) \in \mathbb{R}^{M^2M^2_0(N-1) \times M^2M^2_0} \tag{15}$$

$$\beta_r^x = (I_{M_x} \otimes \alpha_r^x) \in \mathbb{R}^{M^2M^2_0(N-1) \times M^2M^2_0} \tag{16}$$

$$\beta_i^w = (I_{M^2} \otimes \alpha_i^w) \in \mathbb{R}^{M^2M^2_0(N-1) \times M^2M^2_0} \tag{17}$$

$$\beta_r^y = (I_{M^2} \otimes \alpha_r^y) \in \mathbb{R}^{M^2M^2_0(N-1) \times M^2M^2_0} \tag{18}$$

$$\beta_y^r = (I_{M^2} \otimes \alpha_y^r) \in \mathbb{R}^{M^2M^2_0(N-1) \times M^2M^2_0} \tag{19}$$

where $I_k$ denotes an identity matrix of order $k$. Now, using the above transformation matrices we can define the following relationships, $\beta_i^f \beta_r^f$, following relationships. Now, it can easily be noticed that the eigenvectors of $\Omega^f$ consist of the following equation

$$\Psi(f,x,y) = \Psi^f + \Psi^x + \Psi^y = T_R(\Omega^f + \Omega^x + \Omega^y)T_R^{-1}. \tag{20}$$

Now, it can easily be noticed that the eigenvectors of $\Psi(f,x,y)$ are nothing but the transformation matrix $T_R$. Upon estimating this transformation matrix, $\Omega^f, \Omega^x$ and $\Omega^y$ can be estimated with the same permutation order using $\Omega^f = T_R^{-1}\Psi T_R$, $\Omega^x = T_R^{-1}\Psi T_R$ and $\Omega^y = T_R^{-1}\Psi T_R$. The arguments of $\Omega^f, \Omega^x, \Omega^y$ shall provide the triple $\{f_p, \omega_p^x, \omega_p^y\}_{p=1}^{P}$ from which the frequencies and their 2-D DOAs can easily be estimated.

If in addition to the carrier frequencies and DOAs, signal $x(t)$ is required then by assuming $P \leq M + N - 1$, the array manifold matrix can be formed with the estimated parameters and using (8), $S(f)$ can be estimated, from which the signal $x(t)$ can easily be determined.

D. Identifiability and minimum sampling rate

Proposition 3.1: With URA comprising of $M_x \times M_y$ sensors and a $N-1$ channel delay network outlined in Section III-A, and further assuming sources to be uncorrelated, the $P$ carrier frequencies and their 2-D DOAs are recoverable almost surely (assuming no-noise) if

i) $L \leq 1/|B'|

ii) $\omega_{p_1} \neq \omega_{p_2} + m, \omega_{p_1} \neq \omega_{p_2} + m, f_{p_1} \neq f_{p_2},$ for all $1 \leq p_1, p_2 \leq P, p_1 \neq p_2, m \in \mathbb{Z}$

iii) $P \leq \min\{M_x^2M_y^2(N-1), M_x^2(M_y-1)N, (M_x^2-1)M_y^2N\}.$

Due to lack of space, only a brief outline of proof is provided here. As mentioned earlier (refer (5)) that $S_p(f)$ is a periodic spectrum corresponding to the $p$th source with period $f_s = 1/LT$. Since the bandwidth of $s_p(t)$ cannot exceed $B$, in order to avoid aliasing (i.e., $S_p(f + m f_s) \cap S_p(f + (m + 1) f_s) = \emptyset, m \in \mathbb{Z}$), $f_s \geq B$ or $L \leq 1/B'$. For the ESPRIT algorithm, DST manifold matrix $A_{std}$ must be full rank. Since $A_{std}^\dagger A_{std}$ and $A$, are Vandermonde matrices, if the second condition of the proposition is satisfied then by [16, Theorem 3], $A_{std}$ will be full rank almost surely. By observing the row-sizes of the transformation matrices (14) - (19), the maximum number of identifiable parameters can easily be proved.

Now, the first condition implies that a sampling frequency of $f_s \geq B$ would be sufficient and since the configuration consists of $M = M_x M_y$ sensors and $N - 1$ channel delay network, the minimum overall sampling rate $f_{\text{prop}}^{\text{min}} = (M + N - 1)B$ and since $B \ll 1/T, f_{\text{prop}}^{\text{min}} \ll f_{nyq}$. With this minimum sampling rate assuming $M_x$ and $M_y$ to be even and $N < M_x^2M_y^2$, from third condition up to $f_{\text{max}} = M(N-1)/4$, i.e., $O(MN^2)$ carrier frequencies and their 2-D DOAs can be estimated.

Most importantly, the limit provided here is for the extreme case where all the sources exactly alias to the same frequency.

However, when the bands are separated (most often the case in practice), many more carrier frequencies and their DOAs can be estimated by applying the above approach to each individual filtered band. The following section corroborates these results through simulations.

IV. SIMULATION RESULTS

Simulations are performed to test the capability and performance of the proposed approach described in the previous sections. In all our simulations, we assume $F = [0, 5]$ GHz, narrowband signal bandwidth $B = 10$ MHz, the number of sensor elements $M_x = M_y = 3$ and the delay factor $\tau = 0.5$. Further, for all the results presented here, we chose the extreme case where all the carrier frequencies exactly alias onto the same frequency due to sub-Nyquist sampling.

First, we conducted simulations to test the capability at minimum sampling rate as discussed in the previous section. To demonstrate this capability, we assumed $N = 10$ and $f_s = 10$ MHz. We fixed $P = 18$ since upto 18 carrier frequencies and their DOAs can be estimated with the chosen choice of the configuration (see Proposition 3.1). It is important to notice that since $f_s = B = 10$ MHz, all the bands will exactly alias between [0, 10] MHz. A very high SNR of around 40dB was assumed for this simulation and Fig. 3 shows the actual and the estimated frequency and the 2-D DOAs (azimuth and elevation). The figure clearly shows that despite all the bands exactly aliased and $P > M$, the frequency and their 2-D DOAs are estimated with very good accuracy and are close to actual values (note that under noise-less condition the estimation will be exact).

Next, we conducted simulations to test the performance of the proposed approach. For this simulation we chose $P = 12$ and $f_s = 250$ MHz, (i.e., downsampling factor $L = 20$). The carrier frequencies chosen were separated by a factor of 250 MHz, so that after sampling they all alias to the same frequency band. Due to space constraint, instead of providing separate performance plots corresponding to each of frequency and DOAs, since $P < M + N - 1$, the combined
The receiver architecture is modified by adding an additional channel delay network to all the sensors (hardware complexity same as that of [12]) is provided, which also serves as a good lower bound. Note that the same estimation algorithm as outlined in Section III-C2 (without the rank enhancement step) can be used for estimation on this benchmark array. Fig. 4 shows the root mean squared error (RMSE) performance of the proposed approach for different values of \( N \) and the benchmark array. For the benchmark array we have fixed \( M_x = M_y = 3 \) and \( N = 10 \). It is important to note that for this configuration, the benchmark array requires \( M(N-1) \) channels (i.e., 81 channels) and operates on a covariance matrix of dimension \( 90 \times 90 \), whereas the proposed approach requires only \( M + N - 1 \) (\( 9 + N - 1 \)) channels and operates on a much smaller dimension of \( M_x^2 M_y^2 N \times M_x^2 M_y^2 N \) (for this case it is \( 4N \times 4N \)). The performance improvement with increase in \( N \) can clearly be noticed from the figure. In particular, observe that for \( N = 30 \) the performance is very close to the benchmark array. It is important to note the reduction of the overall sampling rate in addition to the reduction of the hardware and computation as mentioned above; while the overall sampling rate with the proposed approach is about 9.5 GHz (for \( N = 30 \)), but with the benchmark array it requires about 20.25 GHz.

V. CONCLUSION

In this paper, a new scheme for joint frequency and 2-D DOA estimation at sub-Nyquist sampling rates using a standard URA comprising of \( M = M_x \times M_y \) is presented. The receiver architecture is modified by adding an \( N - 1 \) delay channel network to one element. By combining the URA and the delay channel network outputs a larger DST array is formed and a sub-space algorithm based on ESPRIT is presented. With this proposed approach, it is shown that one can jointly estimate \( O(MN/4) \) frequencies and their 2-D DOAs. These results are further verified through simulations. The advantage of the proposed scheme (as also demonstrated through simulations) can be leveraged to not only reduce the number of sensors but also to obtain huge savings in computation and in sampling rates.

![Fig. 3. Actual and estimated carrier frequencies and their 2-D DOAs for \( P = 18 \).](image)

![Fig. 4. Comparison of RMSE vs SNR for the reconstructed spectrum for different delay channels and the benchmark array.](image)

REFERENCES