Positive Controllability of Positive Dynamical Systems

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Abstract. The present paper is devoted to a study of constrained controllability and controllability for linear dynamical systems if the controls are taken to be nonnegative. In analogy to the usual definition of controllability it is possible to introduce the concept of positive controllability. We shall concentrate on approximate positive controllability for linear infinite-dimensional dynamical systems when the values of controls are taken from a positive closed convex cone and the operator of the system is normal and has pure discrete point spectrum. The special attention is paid for positive infinite-dimensional linear dynamical systems. Several remarks and comments on the relationships between different concepts of controllability are given.

I. INTRODUCTION

Controllability is one of the fundamental concept in mathematical control theory [1], [3], [6]. Roughly speaking, controllability generally means, that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature there are many different definitions of controllability which depend on class of dynamical system [1], [3], [6], [9], [12], [14], [16]. Problems of controllability for linear control systems defined in infinite-dimensional Banach spaces, have attracted a good deal of interest over the past 20 years. For infinite dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability [1], [3], [6], [7] [12], [13], [14], [15] and [16]. It follows directly from the fact, that in infinite-dimensional spaces there exist linear subspaces which are not closed. Most of the literature in this direction so far has been concerned, however, with unconstrained controllability, and little is known for the case when the control is restricted to take on values in a given subset of the control space. Until now, scarce attention has been paid to the important case where the control of a system are nonnegative. In this case controllability is possible only if the system is oscillating in some sense. Therefore, the most difficult case for constrained controllability is for systems with real eigenvalues [11].

The present paper is devoted to a study of constrained approximate controllability [7], [8], [11] for linear normal infinite-dimensional dynamical systems if the controls are taken to be nonnegative. In analogy to the usual definition of controllability it is possible to introduce the concept of approximate positive controllability [9]. For such dynamical systems direct verification of constrained approximate controllability is rather difficult and complicated [8]. Therefore, we generally assume that the values of controls are taken from a positive closed convex cone [11] and the operator of the system is normal and has pure discrete point spectrum [12], [14]. The special attention is paid for positive infinite-dimensional linear dynamical systems i.e., for dynamical systems preserving positivity [9].

II. NOTATIONS AND SYSTEM DESCRIPTION

In this section we introduce some basic notations and definitions which will be used in the parts of the paper. Throughout this paper we use X to denote infinite dimensional separable real Hilbert space. By \( L^p([0,t],\mathbb{R}^m) \), \( 1 \leq p \leq \infty \) we denote the space of all \( p \)-integrable functions on \([0,t]\) with values in \( \mathbb{R}^m \), and \( L^\infty_p([0,\infty),\mathbb{R}^m) \) the space of all locally \( p \)-integrable functions on \([0,\infty)\) with values in \( \mathbb{R}^m \).

Let us consider linear infinite-dimensional time-invariant control system of the following form

\[
x'(t) = Ax(t) + Bu(t)
\]

(2.1)

Here \( x(t) \in X \) - infinite-dimensional separable Hilbert space which is a vector lattice with a strictly positive linear form.

\( B \) is a linear bounded operator from the space \( \mathbb{R}^m \) into \( X \). Therefore operator \( B=[b_1,b_2,...,b_n,...,b_m] \) and
\[ Bu(t) = \sum_{j=1}^{j=m} b_j u_j(t) \]

where \( b_j \in X \) for \( j = 1, 2, ..., m \), and
\[ u(t) = [u_1(t), u_2(t), ..., u_j(t), ..., u_m(t)]^T. \]

We would like to emphasize that the assumption that linear operator \( B \) is bounded, rules out the application of our theory to boundary control problems, because in this situation \( B \) is typically unbounded.

A: \( X \ni D(A) \to X \) is normal generally unbounded linear operator with compact resolvent \( R(s, A) \) for all \( s \) in the resolvent set \( \rho(A) \). Then operator \( A \) has the following properties \([1], [3], [14], [16]\\):

1) Operator \( A \) has only pure discrete point spectrum \( \sigma_p(A) \) consisting entirely with isolated eigenvalues \( s_i, i = 1, 2, 3, ... \). Moreover, each eigenvalue \( s_i \) has finite multiplicity \( n_c = \infty, i = 1, 2, 3, ... \) equal to the dimensionality of the corresponding eigensubspace.

2) The eigenvectors \( x_{ik} \in D(A), i = 1, 2, 3, ..., k = 1, 2, 3, ..., n_i \), form a complete orthonormal set in the separable Hilbert space \( X \).

3) Operator \( A \) generates an analytic semigroup of linear bounded operators \( S(t) : X \to X \), for \( t \geq 0 \).

Let \( U^+ \subset R^m \) be a positive cone in the space \( R^m \), i.e. \( U^+ = \{ u \in R^m : u_j \geq 0 \ \forall j = 1, 2, ..., m \} \). We define the set of admissible nonnegative controls \( U_{ad} \) as follows
\[ U_{ad} = \{ u \in L^1_{loc}([0,\infty), R^m) : u(t) \in U^+ \ \text{a.e. on} \ [0,\infty) \} \]

It is well known (see e.g. \([1], [3] \) or \([16]\\), that for each \( u \in U_{ad} \) and \( x(0) \in X \) there exists unique so called mild solution \( x(t, x(0), u) \in D(A), t \geq 0 \) of the equation \((2.1)\\) given by
\[ x(t, x(0), u) = S(t)x(0) + \int_0^t S(t-s)Bu(s)ds \]

We say that dynamical system \((2.1)\\) is positive if the semigroup \( S(t) \) and operator \( B \) are positive \([9]\\). In this case the solution \( x(t, x(0), u) \) for initial condition \( x(0) \in X^+ \) and admissible control \( u \in U_{ad} \) remains in \( X^+ \) for all \( t \geq 0 \).

We define the attainable or reachable set in time \( T \) (from the origin) by
\[ K_T(U^+) = \left\{ \int_0^T S(T-s)Bu(s)ds : u \in U_{ad} \right\} \]

The set \( K_\infty(U^+) = \cup_{T=0} K_T(U^+) \) is called the attainable or reachable set in finite time.

Using the concept of attainable set we may define different kinds of controllability for dynamical system \((2.1)\\). Generally, for infinite dimensional dynamical system it is necessary to introduce two fundamental notions of controllability, namely exact (strong) controllability and approximate (weak) controllability. However, since our dynamical system has infinite dimensional state space \( X \) and finite dimensional control space \( R^m \), then by \([13] \) and \([15]\\) it is never exactly controllable in any sense. Therefore, in the sequel we shall concentrate only on approximate controllability with positive controls for system \((2.1)\\).

**Definition 2.1.** \([1], [3], [6]\\) Dynamical system \((2.1)\\) is said to be approximately controllable with nonnegative controls if \( \text{cl} K_\infty(U^+) = X \)

In the unconstrained case, i.e. when the controls values are taken from the whole space \( R^m \), we say simply about approximate controllability of system \((2.1)\\).

The above notion of approximate controllability is defined in the sense that we want to reach a dense subspace of the entire state space. However, in many instances for positive systems with nonnegative controls, it is known that all states are contained in a closed positive cone \( X^+ \) of the state space. In this case approximate controllability in the sense of the above definition is impossible but it is interesting to know conditions under which the reachable states are dense in \( X^+ \). This observation leads to the concept of so-called positive approximate controllability.

**Definition 2.2.** \([9]\\) Dynamical system \((2.1)\\) is said to be approximately positive controllable if \( \text{cl} K_\infty(U^+) = X^+ \).

**Remark 2.1.** From the above two definitions directly follows, that approximate controllability with nonnegative controls always implies approximate positive controllability. However, the converse statement is not generally true.

Finally, we shall recall some fundamental theorems concerning unconstrained and constrained approximate controllability of dynamical system \((2.1)\\). In order to do that let us introduce the following notations.

Using eigenvectors \( x_{ik}, i = 1, 2, 3, ..., k = 1, 2, 3, ..., n_i \), we introduce for the operator \( B \) the following notation \([6]\\):
\[ B_i = \left( \langle b_j, x_{ik} \rangle \right)_x \]

For each \( i \in \{1, 2, 3, ..., n_i \} \) and \( k \in \{1, 2, 3, ..., n_i \} \), we denote \( b_{ik} = \langle b_j, x_{ik} \rangle_x \).

For simplicity of notation let us denote \( b_{ij} = \langle b_j, x_{ik} \rangle_x \) for \( i = 1, 2, 3, ..., n_i \) and \( j = 1, 2, 3, ..., m \). Therefore, we may express matrices \( B_i \) and vectors \( b_i \) in a more convenient form as follows.

\[ b_{ij} = \langle b_j, x_{ik} \rangle_x \]

for \( i = 1, 2, 3, ..., n_i \) and \( j = 1, 2, 3, ..., m \). Therefore, we may express matrices \( B_i \) and vectors \( b_i \) in a more convenient form as follows.
Therefore, dynamical system (2.1) with one scalar nonnegative control is never approximately controllable [11]. Moreover, it should be stressed, that in general case for multiple eigenvalues, it is not so easily to verify the hypothesis that the set of given vectors forms a positive basis in the Euclidean space.

**Remark 2.3** Using the concept of polar cone $C^0$, the results stated in above theorem can be extended for constrained controls which take their values from a given closed cone $C$ with nonempty interior $\text{int}C \subseteq U_{ad}$ [11].

### III. POSITIVE CONTROLLABILITY

In this section we shall recall the results concerning approximate positive controllability for dynamical systems (2.1). We start with the following negative result on approximate positive controllability.

**Theorem 3.1** [7] If there exists $p$ and $q$ such that eigenvalue $s_p \in \mathbb{R}$ and coefficients $b_{pq}$ have the same sign for every $j=1,2,...,m$, then the dynamical system (2.1) is not approximately positive controllable.

From Theorem 3.1 and Remark 2.1 follows the next result concerning approximate controllability of system (2.1) with nonnegative controls.

**Corollary 3.1.** [7] If the assumptions of Theorem 3.1 are satisfied, then the dynamical system (2.1) is not approximately controllable with nonnegative controls.

### IV. POSITIVE STATIONARY PAIRS

In section 3 we have obtained some negative results concerning approximate positive controllability for dynamical system (2.1). However, it is often not so important to reach the entire positive cone of the state space. It suffices to steer approximately dynamical system to particular positive states and held constant by a nonnegative control for all times. This observation directly leads to the concept of so called positive stationary pairs [9]. In this section we generally assume that the dynamical system (2.1) is positive in the sense stated in section 2.

**Definition 4.1** [9] We call a pair $\{x_0, u_0\} \in (X^\ast \setminus \{0\}) \times U^+$ positive stationary pair if $Ax + Bu = 0$. In this case $x(t, x_0, u_0) = x_0 \in X^+$ is a nonzero constant solution of the equation (2.1) for $t \geq 0$, $u(t) = u_0$ and $x(t) = x(t, x_0, u_0) > 0$.

**Theorem 4.1** [9] Let dynamical system (2.1) be positive and $S(t)$ be uniformly exponentially stable positive semigroup. Then to each $u_0 \in U^+ \setminus \ker B$ there exists exactly one $x_0 = -A^\dagger B u_0$ such that $\{x_0, u_0\}$ is a positive stationary pair. Moreover, if $\{x_0, u_0\}$ is a positive stationary pair, and we choose $x(0) \in X^+$ and $u(t) = u_0$,
\( t \geq 0 \), then the solution of the equation (2.1) tends to \( x \) as \( t \to \infty \).

**Corollary 4.1** Let \( \text{Re}(s_i) < 0 \). Then to each \( u_t \in U^\ast \ker B \) there exists exactly one

\[
x = \sum_{j=1}^{\infty} s_j^{-1} \sum_{k=1}^{j} x_k, \sum_{j=1}^{m} b_j u_j \| X \]

(4.1)

such that \( \{x, u_t\} \) is a positive stationary pair.

**Proof.** Since the spectrum of the linear operator \( \sigma(A) \) is pure discrete point spectrum, we conclude that the inequality \( \text{Re}(s_i)<0 \) is a necessary and sufficient condition for so called uniform stability of linear dynamical system [1], [9]. Therefore, using general spectral formula for the operator \( A^{-1} \) and Theorem 4.1 stated above we obtain immediately equality (4.1).

**Remark 4.1** Many valuable remarks and comments on the relationships between different kinds of stability (uniform exponential stability, strong stability, weak stability) of the linear abstract differential equation (2.1) and the existence of positive stationary pairs for positive dynamical systems can be found in the paper [9].

V. CONCLUSIONS

The present paper contains several results on constrained controllability for linear infinite-dimensional selfadjoint dynamical systems. Special attention is paid on positive dynamical systems. Using spectral properties of normal generally unbounded linear operators with pure discrete point spectrum, conditions for different kinds of constrained controllability have been formulated and proved. General results have been also applied for constrained controllability considerations for linear distributed parameter dynamical systems described by linear partial differential equations of parabolic type with various kinds of boundary conditions.

Some kinds of the presented results can be extended to cover the case of infinite-dimensional normal dynamical systems with discrete and continuous spectrum. It is also possible to extend the result for second-order infinite-dimensional dynamical systems.

REFERENCES.