Constrained Controllability of Distributed Parameter System
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Abstract — In the paper constrained controllability of distributed parameter dynamical system defined in infinite-dimensional domain is considered. Using spectral theory of unbounded differential operators, necessary and sufficient conditions for constrained approximate controllability are formulated and proved. Remarks and comments on the relationships between different kinds of controllability are also given.

I. INTRODUCTION

Controllability is one of the fundamental concept in mathematical control theory [1], [3], [6]. Roughly speaking, controllability generally means, that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In literature there are many different definitions of controllability which depend on class of system and on the form of admissible controls [1], [3], [6], [9], [11], [13], [15].

Problems of controllability for linear control systems defined in infinite-dimensional Banach spaces, have attracted a good deal of interest over the past 30 years. For infinite dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability [1], [3], [6], [11], [12], [13], [14] and [15]. It follows directly from the fact, that in infinite-dimensional spaces there exist linear subspaces which are not closed.

Most of the literature in this direction so far has been concerned, however, with unconstrained controllability, and little is known for the case when the control is restricted to take on values in a given subset of the control space. Until now, scare attention has been paid to the important case where the control of a system are nonnegative. In this case controllability is possible only if the system is oscillating in some sense.

The present paper is devoted to a study of constrained approximate controllability for linear infinite-dimensional distributed parameter dynamical systems with constrained controls. For such dynamical systems direct verification of constrained approximate controllability is rather difficult and complicated [8]. Therefore, we shall concentrate on special case, when the values of admissible controls are taken from a given closed convex cone [10], more precisely we shall consider the cone of nonnegative controls. Using the general results given in the survey paper [6] we shall present without proofs necessary and sufficient conditions for approximate controllability with nonnegative controls. The present paper extends for constrained controls the results given in [2] for unconstrained controls.

II. SYSTEM DESCRIPTION

Let us consider distributed parameter dynamical system described by the following linear partial differential equation defined on infinite domain [2], [7]

\[ v_t(z,t) = A_k v(z,t) + b_1(z) u_1(t) + b_2(z) u_2(t) \] (1)

with initial condition

\[ v(z,0) \in L_2(R) \] (2)

where \( z \in R \) and \( t \geq 0 \), \( b_1(z) \in L_2(R) \), \( b_2(z) \in L_2(R) \), and \( k \) is an integer number.

In the next section we shall also consider dynamical system of the form (2.1) but with only one scalar control (i.e. \( b_2(z) = 0 \)).

In the sequel it is generally assumed, that the admissible controls \( u_1(t) \in L_2([0,t], R^m) \), and \( u_2(t) \in L_2([0,t], R^s) \).

\( A_k : D(A_k) \rightarrow L_2(R) \) is a linear unbounded differential operator defined as follows

\[ D(A_k) = \{ v(z) \in L_2(R) : A_k v(z) \in L_2(R) \} \] (3)

\[ A_k v(z) = v_{zz}(z) + (k - z^2) v(z) \] (4)

Now, for the convenience, let us collect some well known facts about the operator \( A_k \) [2], [6], [7].

Operator \( A_k \) is selfadjoint with compact resolvent and is an infinitesimal generator of an analytic semigroup of linear bounded operators \( S_k(t) : L_2(R) \rightarrow L_2(R) \), for \( t \geq 0 \).

Moreover, operator \( A_k \) has only pure discrete point spectrum \( \sigma(A_k) = \{ s_{kn} \} \), where the eigenvalues:

\[ s_{kn} = -2n + k - 1 \text{, for } n = 0, 1, 2, \ldots \]

are all of multiplicity one.

The corresponding eigenfunctions

\[ g_k(z) = (2^n n!)^{-0.5} (\pi)^{0.25} \text{exp}(-0.5z^2) H_k(z) \quad n = 0,1,2,\ldots \]

where

\[ H_k(z) = (-1)^{k}\text{exp}(z^2) d^k/dz^k(\text{exp}(-z^2)) \]

are Hermite’s polynomials, and form a complete orthonormal system in a separable Hilbert space \( L_2(R) = V \).

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It is well known [2], [3], [7], that abstract ordinary differential equation (1) with initial conditions \( v(z,0) \in D(A) \) has for each \( t_1 > 0 \) an unique solution \( v(t;v(z,0),u) \) such that \( v(t) \in D(A) \) for \( t \in (0, t_1] \).

**Definition 2.1.** [1], [3], [6]. Dynamical system (1) is said to be approximately controllable with nonnegative controls if for any initial condition \( v(z,0) \in V \), any given final condition \( v_T \in V \), and each positive real number \( \varepsilon \) there exist a finite time \( t_1 < \infty \) (depending generally on \( v(z,0) \) and \( v_T \)) and admissible controls \( u_1(t) \in L^2([0,t_1],R^r) \) and \( u_2(t) \in L^2([0,t_1],R^r) \) such that

\[
\left\| v(t_1; v(z,0), u_1, u_2) - v_T \right\| \leq \varepsilon
\]

The above notion of approximate controllability is defined in the sense that we want to reach a dense subspace of the entire state space. However, in many instances for systems with restrictions on the controls, it is known that all states are contained in a closed positive cone \( V^+ \) of the state space. In this case approximate controllability in the sense of the above definition is impossible but it is interesting to know conditions under which the reachable states are dense in \( V \). This observation leads to the concept of so-called positive approximate controllability.

**Definition 2.2.** [9] Dynamical system (1) is said to be positively approximately controllable if for any initial condition \( v(z,0) \in V^+ \), any given final condition \( v_T \in V^+ \), and each positive real number \( \varepsilon \) there exist a finite time \( t_1 < \infty \) (depending generally on \( v(z,0) \) and \( v_T \)) and admissible controls \( u_1(t) \in L^2([0,t_1],R^r) \) and \( u_2(t) \in L^2([0,t_1],R^r) \) such that

\[
\left\| v(t_1; v(z,0), u_1, u_2) - v_T \right\| \leq \varepsilon
\]

From the above definitions directly follows, that approximate controllability with nonnegative controls always implies positive approximate controllability.

### III. CONSPRAINED CONTROLLABILITY

Now, let us formulate several results concerning constrained approximate controllability of dynamical system (1).

**Theorem 3.1.** Dynamical system (1) is approximately controllable with nonnegative controls if and only if

\[
b_{1n} b_{2n} < 0 \quad \text{for every } n=0,1,2,...
\]

where

\[
b_{jn} = \left\langle b_j(z), g_n(z) \right\rangle = \int_{V} b_j(z) g_n(z) dz > 0
\]

for \( j=1,2 \) and every \( n=0,1,2,... \)

Proof of the above theorem is based on general necessary and sufficient constrained controllability conditions given in the paper [6].

**Corollary 3.1.** [2] Dynamical system (1) is approximately controllable (with unconstrained controls) if and only if

\[
b_1^2 + b_2^2 \neq 0 \quad \text{for every } n=0,1,2,...
\]

In other words dynamical system (1) is approximately controllable (with unconstrained controls) if and only if \( b_{1n} \neq 0 \) or \( b_{2n} \neq 0 \) for every \( n=0,1,2,... \).

Therefore, approximate controllability with unconstrained controls may occur even for one scalar controls, which is impossible for approximate controllability with nonnegative controls.

**Corollary 3.2.** [2] Let \( b_2(z) = 0 \). Then system (1) is approximately controllable (with unconstrained controls) if and only if

\[
b_{1n} \neq 0 \quad \text{for every } n=0,1,2,...
\]

In the next part of this section it is assumed that \( b_2(z) = 0 \), i.e. there is only one positive scalar control \( u(t) \in R^r \).

**Theorem 3.2.** Dynamical system (1) is not positively approximately controllable.

In practice, it is often not so important to reach approximately the entire positive cone \( V^+ \) of the state space \( V \). Sometimes it suffices to reach approximately by nonnegative controls only particular positive states in the positive cone \( V^+ \). This observation leads directly to the concept of so-called positive stationary pairs.

**Definition 3.3.** [9] A pair \( \{v_s,u_s\} \in V^+ \times R^r \) is said to a positive stationary pair for dynamical system (2.1) if \( A_s v_s + b_s u_s = 0 \).

Let us observe, that if \( \{v_s,u_s\} \) is a positive stationary pair, then \( v(z,t) = v_s \) is a nonzero constant solution of (2.1) for \( u(t) = u_s \) and \( v(z,0) = v_s \). Moreover, there exists strong connection between existence of positive stationary pairs and stability of dynamical system (2.1).

**Theorem 3.3.** Let \( -2n+k-1 \neq 0 \). Then to each \( u_1 \in R^r \) there exists exactly one \( v_s \in V^+ \) such that \( \{v_s,u_s\} \) is a positive stationary pair.

### V. CONCLUSION

In the paper approximate constrained controllability for linear infinite dimensional dynamical system has been considered. Using methods of functional analysis, specially theory of linear unbounded differential operators and linear semigroups, necessary and sufficient conditions for approximate controllability have been formulated and proved.

The obtained results can be extended to other types of infinite dimensional distributed parameter dynamical systems described by linear partial differential equations. The present paper extends for constrained controls the results given in [2] for unconstrained controls.
REFERENCES