**Stabilization and Trajectory Tracking in Discrete-Time of an Autonomous Four Rotor Mini-Rotorcraft**

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**Abstract:** In this paper, we present a stabilization and trajectory tracking controller in discrete–time for an autonomous four rotor mini-rotorcraft. The control algorithm proposed is based on the feedback linearization input-output for nonlinear discrete-time systems. The dynamic model of the autonomous four rotor rotorcraft is obtained via Lagrange formalism. Simulation results are presented to illustrate the performances of this controller.

**Introduction**

The miniature and autonomous flying robots arouse a growing interest in the civil and military domains; the fields of application of these vehicles are vast. We can state the ecological exploration mission and air cartography, search and rescue, surveillance and remote inspection. In this paper, we are particularly interested in controlling a mini rotorcraft having four rotors. Advantages of using a multi-rotor helicopter are the increased payload capacity and high manoeuvrability and the fact that the two motors turn in the clockwise direction whereas the two other turn in the opposite clockwise direction, gyroscopic effects and aerodynamic torques tend to cancel in trimmed flight. Disadvantages are the increased helicopter weight and increased energy consumption due to the extra motors.

A control strategy based on the backstepping techniques proposed in [4] for configuration stabilization of quasi-stationary flight conditions of a four rotor vertical take-off and landing (VTOL). The stabilization problem of a four rotor rotorcraft is also studied and tested in [2] where the nested saturation control algorithm is used. In [5], flatness and motion planning are combined to solve the point to point control problem with a predefined path of the four rotor rotorcraft.

In this paper we present a discrete-time controller design of a four rotor helicopter modelled via Lagrange approach, the control strategy is based on the feedback linearization input-output for nonlinear discrete-time systems using the Euler approximate discrete-time model of the Lagrangian model when the sampling period should typically be sufficiently small in order to get a good approximation of the exact discrete-time model. Discrete-time designs are important because most controllers are implemented using digital computers with A/D and D/A converters (simpler and zero-order hold) and the digital controllers based on the Euler approximate discrete-time model may outperform discretized continuous-time controllers.

The paper is organized as follows: Section II presents the dynamical model of the rotorcraft. Section III gives the controller synthesis. The simulation results are shown in section IV. The conclusions are finally given in section V.

**II. Dynamic model of autonomous rotorcraft**

In this section, a Lagrangian model is derived for the four rotor helicopter in generalized coordinates [1, 2, 6]. Consider figure 1. Let $\mathbb{R}_O = (O, e_x, e_y, e_z)$ denote an inertial frame. Let $\mathbb{R}_G = (G, e_1, e_2, e_3)$ denote a body fixed frame attached to the helicopter. The position of the center of mass of the rotorcraft relative to the frame $\mathbb{R}_O$ is denoted $\xi = (X, Y, Z)$ and $\eta = (\psi, \theta, \phi)$ are the Euler angles (yaw, pitch and roll).

![Figure 1. Frames of the four rotor rotorcraft with thrust inputs.](image)

The total thrust produced by the four rotors is given by: $u_i = f_1 + f_2 + f_3 + f_4$ where $f_i = b_i \omega_i^2$ is the thrust generated by the rotor $i = 1, 2, 3, 4$ and $\omega_i$ is the angular speed generated by the motor $M_i$ and $b_i > 0$ is a parameter. The torques acting of the four rotors helicopter result from the action of the thrust forces difference of each pair of rotors are denoted by:
\[ \tau = (\tau_\psi, \tau_\theta, \tau_\phi) \] around axes \( e_1, e_2 \) and \( e_3 \) respectively, with \( \tau_\phi = (f_2 - f_4)d \), \( \tau_\theta = (f_3 - f_1)d \) and \( \tau_\psi = (f_2 + f_4 - f_1 - f_3)\kappa \), where \( d \) represents the distance from the rotors to the center of gravity of the helicopter and \( \kappa > 0 \) is a constant.

In the sequel we use the state space representation with the following notations:
\[
\begin{align*}
x^1 &= \xi = [x_1 \ x_2 \ x_3]^T, \\
x^2 &= \zeta = [x_2 \ x_4 \ x_5]^T, \\
x^3 &= \eta = [x_7 \ x_9 \ x_{11}]^T, \\
x^4 &= \hat{\eta} = [x_8 \ x_{10} \ x_{12}]^T.
\end{align*}
\]
Let \( x = (x^1, x^2, x^3, x^4) \) denote the state variable, \( u = (u_1, \tau) \) denote the control input and \( y = (y_1, y_2) = (x^1, x^3) \) denote the output of the rotorcraft.

The Euler discretization of Lagrangian model [6] using the state space representation gives:
\[
\begin{align*}
x^1(k+1) &= x^1(k) + Tx^2(k) \\
x^2(k+1) &= x^2(k) + T\Lambda(x^1(k), u_1(k)) \\
x^3(k+1) &= x^3(k) + Tx^4(k) \\
x^4(k+1) &= x^4(k) + TJ^{-1}(x^3(k)) \Gamma(x^3(k), x^4(k)) + TJ^{-1}(x^3(k))\tau(k)
\end{align*}
\]

(1)

Where \( \Gamma(\eta, \hat{\eta}) \) is the Coriolis and Centrifugal vector, \( J(\eta) \) is the inertia matrix for the rotorcraft expressed directly in terms of the generalized coordinates \( \eta \), \( k = 0, 1, 2, \ldots; \ T > 0 \) is the sampling time and
\[
\Lambda(x^1, u_1) = \frac{1}{m} \begin{pmatrix}
cos x_7 & x_5 & \cos x_{11} + \sin x_7 & \sin x_7 & \cos x_{11} - \cos x_7 & \sin x_7 & \sin x_7 & \cos x_{11} \\
\cos x_7 & \sin x_7 & \cos x_{11} - \cos x_7 & \sin x_7 & \cos x_{11} + \sin x_7 & x_5 & \cos x_{11} & \sin x_7
\end{pmatrix} u_1
\]

III. Discrete-time control design

In this section, we use a feedback linearization input-output technique in discrete-time for stabilization and tracking a desired trajectory of the helicopter. This technique can be treated more or less similarly as for continuous-time nonlinear system [3].

Since the dynamic model of the rotorcraft (1) is under-actuated (four inputs and six outputs), we are interested to control the translational position \( \xi = (X, Y, Z) \) and the yaw angular position \( \psi \). Let \( \bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4) = (\xi, \psi) \) denote the outputs that we want to control. The control objective is to find a feedback control \( u = (u_1, \tau) = \alpha(x, v) \) where \( v \) represents a vector of new control such that the closed loop dynamic is input-output decoupled and described by the linear system \( \bar{y}_j(k + \ell_j + 1) = v_j(k), \ j = 1, 2, 3, 4 \). The number \( \ell_j \) is such that \( \bar{y}_j(k + \ell) \) does not depend on the control input \( u \) for \( \ell = 0, 1, 2, \ldots, \ell_j \) and \( \bar{y}_j(k + \ell_j + 1) \) depends explicitly on the control input \( u \) with \( (4 + \sum_{j=1}^{4} \ell_j) = 12 \). The trajectory tracking problem is solved by the linear state feedback \( v(k) \).

As a first step, the dynamics of rotation \( (x^3, x^4) \) can be input-output linearized by the following decoupling feedback laws:
\[
\tau(k) = \Gamma(x^3(k), x^4(k)) + J(x^3(k))\bar{\tau}(k)
\]

(2)

where \( \bar{\tau} \) is the new angular moment defined by:
\[
\bar{\tau} = (\bar{\tau}_\psi, \bar{\tau}_\theta, \bar{\tau}_\phi).
\]

1. Control of the yaw angular position \( \psi \)

The yaw-dynamic can be controlled without difficulties and independently of other movements.

Consider the following subsystem:
\[
\begin{align*}
x_\gamma(k+1) &= x_\gamma(k) + Tx_\gamma(k) \\
x_\gamma(k+1) &= x_\gamma(k) + T\bar{\tau}_\psi(k) \\
\bar{y}_4(k) &= x_\gamma(k)
\end{align*}
\]

(3)

A simple calculation gives:
\[
\bar{y}_4(k + 2) = \Phi_{41}(x(k)) + T^2\bar{\tau}_\psi(k)
\]

where \( \Phi_{41}(x(k)) = x_\gamma(k + 1) + Tx_\gamma(k) \)

Now let us consider the local change of coordinates
\[
z^4(k) = [z^4_1(k) \ z^4_2(k)]^T = [\bar{y}_4(k) \ \bar{y}_4(k + 1)]^T
\]

The subsystem (3) can be written in the following canonical form:
\[
\begin{align*}
z^4(k + 1) &= A_4 z^4(k) + B_4 v_4(k) \\
\bar{y}_4(k) &= z^4_1(k)
\end{align*}
\]

(4)

where \( A_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( B_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

Then the yaw angular position can be controlled by applying the controller:
\[
\bar{\tau}_\psi(k) = \frac{1}{T^2} [ -\Phi_{41}(x(k)) + v_4(k) ], \quad \text{where}
\]
\[
v_4(k) = -L_4(z^4(k) - z^4_{ref}(k)) + \bar{y}_{4ref}(k + 2),
\]

where \( L_4 \) is such that the eigenvalues of the matrix \( (A_4 - B_4 L_4) \) are strictly inside the unit disk \( D(0,1) \).

1. Control of the vertical position \( Z \)
The control input of the vertical position can be obtained considering the following subsystem:
\[
\begin{aligned}
    &x_5(k+1) = x_5(k) + T x_6(k) \\
    &x_6(k+1) = x_6(k) + \frac{T}{m} \cos x_9(k) \cos x_{11}(k) u_1(k) - T g
\end{aligned}
\]
\[
\tilde{y}_1(k) = x_5(k)
\]
We have:
\[
\tilde{y}_1(k+2) = \Phi_{31}(x(k)) + \Phi_{32}(x(k)) u_1(k)
\]
where \( \Phi_{31}(x(k)) = x_5(k) + 2T x_6(k) - T^2 g \) and
\[
\Phi_{32}(x(k)) = \frac{T^2}{m} \cos x_9(k) \cos x_{11}(k).
\]
The subsystem (5) can be transformed to:
\[
\begin{aligned}
    &z^3(k+1) = A_4 z^3(k) + B_3 v_3(k) \\
    &\tilde{y}_3(k) = z^3_1(k)
\end{aligned}
\]
where
\[
z^3(k) = [z^3_1(k) \ z^3_2(k)]^T = [\tilde{y}_3(k) \ \tilde{y}_3(k+1)]^T,
\]
\(A_3 = A_4\) and \(B_3 = B_4\). The controller \(u_1(k)\) is given as follows:
\[
u_1(k) = \frac{1}{\Phi_{32}(x(k))} [-\Phi_{31}(x(k)) + v_3(k)]
\]
where
\[
v_3(k) = -L_3 (z^3(k) - z^3_{\text{ref}}(k)) + \tilde{y}_{3\text{ref}}(k+2)
\]
\(L_3\) is such that the spectrum \(sp(A_4 - B_4 L_3) \subset D(0,1)\).
We can define a function \(r(k)\) such that
\[
u(k) = \frac{r(k)}{\Phi_{32}(x(k))}, \ \forall(x_9, x_{11}) \in ]-\pi/2, \pi/2[\]
where \(r(k) = -\Phi_{31}(x(k)) + v_3(k)\).
In the sequel, the dynamical expressions of the collective input \(u_1(k)\), \(u_1(k+1)\) and \(u_1(k+2)\) are used to control the displacements in the horizontal plane via \(r(k), r(k+1)\) and \(r(k+2)\).

3. Control of the horizontal positions \(X\) and \(Y\)
Introducing the expression of \(u_1(k)\) into system (1) and considering the remaining subsystem
Like previously, we have:
\[
\tilde{y}_1(k+4) = \Phi_{11}(x(k)) + [\cos x_7(k+2) \tan x_9(k+2) + \sin x_7(k+2) \tan x_{11}(k+2)] r(k+2),
\]
where
\[
\Phi_{11}(x(k)) = x_1(k+3) + T x_2(k+1) + [\cos x_7(k+1) \times \tan x_9(k+1) + \sin x_7(k+1) \tan x_{11}(k+1)] r(k+1),
\]
Using a similar procedure, we get:
\[
\tilde{y}_2(k+4) = \Phi_{21} (x(k)) + [\sin x_7(k+2) \tan x_9(k+2) - \cos x_7(k+2) \tan x_{11}(k+2)] r(k+2),
\]
where
\[
\Phi_{21}(x(k)) = x_3(k+3) + T x_4(k+1) + [\sin x_7(k+1) \times \tan x_9(k+1) - \cos x_7(k+1) \tan x_{11}(k+1)] r(k+1).
\]
We obtain the following linear system:
\[
\begin{aligned}
    &\begin{bmatrix} z^3(k+1) \\ \tilde{y}_1(k) \end{bmatrix} = A_4 \begin{bmatrix} z^3(k) \\ \tilde{y}_1(k) \end{bmatrix} + B_3 v_3(k) \\
    &\begin{bmatrix} \tilde{y}_1(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
    &A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ i = 1,2 .
\end{aligned}
\]
Using the fact that:
\[
x^3(k+2) = x^3(k+1) + T x^4(k) + T^2 \tilde{r}(k)
\]
The pitch controller \(\tilde{r}_\varphi\) and the roll controller \(\tilde{r}_\eta\) are given as follows:
\[
\begin{aligned}
    &\tilde{r}_\varphi(k) = \frac{1}{T^2} [-x_9(k+1) - T x_{10}(k) + \varphi_1(k)], \\
    &\tilde{r}_\eta(k) = \frac{1}{T^2} [-x_{11}(k+1) - T x_{12}(k) + \varphi_2(k)],
\end{aligned}
\]
where
\[
\varphi_1(k) = \text{arctan} \left[ \frac{-\Phi_{11}(x(k)) + v_1(k) \cos x_7(k+2) + \Phi_{21}(x(k)) + v_2(k) \sin x_7(k+2)}{r(k+2)} \right],
\]
\[
\varphi_2(k) = \text{arctan} \left[ \frac{-\Phi_{11}(x(k)) + v_1(k) \sin x_7(k+2) + \Phi_{21}(x(k)) + v_2(k) \cos x_7(k+2)}{r(k+2)} \right],
\]
\(v_1(k) = -L_3 (z^3(k) - z^3_{\text{ref}}(k)) + \tilde{y}_{3\text{ref}}(k+4)\) and \(L_3\) is such that the spectrum \(sp(A_4 - B_4 L_3) \subset D(0,1)\); \(i = 1,2\).

IV. Simulation results
The first simulation concerns the stabilization of the rotorcraft and the second simulation concerns the problem of tracking a trajectory. The physical parameters used for the dynamic model of the rotorcraft are:
\[
m = 0.52, \ d = 0.205, \ g = 9.8, \ T = \frac{1}{14}.
\]
The gain vectors of the controller are set to:
\[ L_1 = [0.1430\quad -0.9301\quad 2.2691\quad -2.4600], \]
\[ L_2 = [0.0497\quad -0.4215\quad 1.3391\quad -1.8900], \]
\[ L_3 = [0.6240\quad -1.5800], \]
\[ L_4 = [0.3300\quad -1.1500]. \]

The initial state vector is \( x = 0 \).

Figure 2 (a) shows the evolution of the rotorcraft in 3D and 2D space, and figure 2 (b) shows the time evolution of the actual position and orientation of the rotorcraft in the case of stabilization. The desired position \( \xi_d \) and \( \psi_d \) are:
\[ \xi_d = [30.10^{-2}\quad 30.10^{-2}\quad 80.10^{-2}]^T \]
and \[ \psi_d = 45.\pi/180. \]

Figure 3 (a) depicts the desired and actual trajectories in 3D and 2D space, and figure 3 (b) shows the time evolution of the position and orientation of the rotorcraft in the case of trajectory tracking. Let's note that on these trajectories, we also fixed the desired corresponding velocities. The desired trajectory is chosen as the following:
\[ \psi_{ref}(k) = 45.\pi/180 \quad \text{and} \quad \xi_{ref}(k) = (0.01k + 0.12, 0.01k - 0.12, 0.02k + 0.15) \]

Figures 2 and 3 show that the controlled variables converge towards the desired trajectories.

V. Conclusions

In this paper, we have presented a stabilization and trajectory tracking for an autonomous four rotor helicopter in discrete-time. The dynamic model was obtained using the Euler-Lagrange equation formalism. The proposed control algorithm is based on the feedback linearization input-output for nonlinear discrete-time systems. The simulation results show the performances of the proposed controller.

References