Parametric Identification of Non Linear Systems Using Higher Order Statistics and Communication Signals

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Abstract—in this paper, a new relationship linking cumulants and the coefficients of non linear systems is presented using non Gaussian signal input. This relationship is used to develop a new algorithm based on fourth order cumulants for the identification of the kernels of non linear systems in noiseless and noise environment case. The simulation results are presented to illustrate the performance of proposed algorithm.

I. INTRODUCTION

The quadratic systems constitute an interesting class of second order Volterra systems, where the second order homogeneous Volterra kernel is diagonal [1]. Quadratic models have been successfully used to represent non linear systems in a number of practical applications in the areas of chemical processes, biological systems, and communication and control [1]. They have this form: 

\[ y(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h(i, j)x(t-j)x(t-i) \]

where \( y(t) \) is the system output and \( x(t) \) is the system input. The finite extent of the quadratic systems, which is a special case of Eq. (1), is given by the following formula:

\[ y(t) = \sum_{i=0}^{q} \sum_{j=0}^{q} h(i, j)x(t-j)x(t-i), \quad h(0,0) = 1 \]

(2)

The integer \( q \) constitutes the order of the system. The input signal \( x(t) \) is a stationary zero mean non-gaussian white stochastic process with variance \( \sigma_x^2 = 1 \).

Blind identification of non linear systems (quadratic) is concerned with the determination of the quadratic kernels \( h(i, j) \) on the basis of output information only. In broad terms, blind identification is of interest when access to the excitation input is not available. The complexity of the problem arises from the fact that the output statistics depend nonlinearily on the kernels even if the system is linear [2-5]. This is in sharp contrast with conventional system identification whereby the use of crosscumulants between the output and the input copies depend linearly on the Volterra kernels [6]. Little is known about blind identification of general Volterra systems [2-5]. The solution of the blind identification problem for the special case of a quadratic systems presented in this paper will hopefully provide useful insight in the general cases. So, we are confronted with blind identification and non-gaussianity in which we suppose some statistical property of the input signal and we address the problem of blind identification of quadratic non-linear systems using higher order statistics.

II. PRELIMINARIES AND PROBLEM STATEMENT

In this section, using the relationship proposed by Stoïoglou and McLaughlin [7] in the linear case and we based on equation:

\[ P_{r,m} = \gamma_{m,s} \sum_i h(i)h(j) \prod_{k=1}^{r} h(i + \tau_k) \prod_{k=1}^{s} h(j + \alpha_k) \]

\[ \times \prod_{k=s+1}^{m-1} h(i + j + \beta_k) \]  

(3)

Where \( s \) is an arbitrary integer number satisfying: \( 1 \leq s \leq m-2 \) and \( P_{r,m} = \gamma_{m,s}Z_{r,m} \) (see [8]).

With

\[ Z_{r,m} = \sum_i h(i)h(j) \prod_{k=1}^{r} h(i + \tau_k) \prod_{k=1}^{s} h(j + \alpha_k) \]

\[ \times \prod_{k=s+1}^{m-1} h(i + j + \beta_k) \]  

(4)

\( m \) is the order of cumulants and \( \gamma_{m,s} \) represents the mth order cumulants of the excitation signal at origin.

Based on Eq. (3), the relationship in the non linear case:

\[ P_{r,m} = \gamma_{m,s} \sum_i h(i)h(j) \prod_{k=1}^{r} h(i + \tau_k, l + \tau_k) \]

\[ \times \prod_{k=1}^{r} h(j + \alpha_k, p + \alpha_k) \prod_{k=s+1}^{m-1} h(l + \tau_k, p + \beta_k) \]  

(4)

Changing the order of summation in (4) yields to:

\[ P_{r,m} = \sum_{i,j} h(i)h(j) \prod_{k=1}^{r} h(i + \tau_k, l + \tau_k) \gamma_{m,s} \sum_{j,p} h(j, p) \]
\[ \times \left( \prod_{k=1}^{r} h(j + \alpha_k, p + \alpha_k) \right) \prod_{k=1}^{m-1} \frac{h(i+j + \beta_k, l+p + \beta_k)}{i+j + \beta_k, l+p + \beta_k} \] 

The second order cumulant (AutoCorrelation Function ACF) of the processes \( y(t) \) is described by the following expression \([9]\):

\[ C_{2y}(\tau) = \gamma_{4y} \sum_{i=0}^{\tau} h(i, i) h(i + \tau, i + \tau) \]  

The 3rd order cumulant of the signal \( y(t) \) is given by the following expression \([9]\):

\[ C_{3y}(\tau_1, \tau_2) = \text{Cum}_3[y(t) y(t+\tau_1) y(t+\tau_2)] \]

\[ = \gamma_{6y} \sum_{i=0}^{\tau} h(i, i) h(i + \tau_1, i + \tau_1) h(i + \tau_2, i + \tau_2) \]  

(7)

From Eqs (6) and (7) we obtain the mth order cumulant of the output signal \( y(k) \):

\[ C_{mk}(\tau_1, \ldots, \tau_m) = \gamma_{2mn} \sum_{i=0}^{q} h(i, j) h(i + \tau_1, i + \tau_1) \cdots h(i + \tau_m, i + \tau_m) \]  

(8)

if we suppose \( l = i \) and \( p = j \) in Eq.(4) we obtain:

\[ P_{r,m} = \sum_{i} h(i, i) \left( \prod_{k=1}^{r} h(i + \tau_k, i + \tau_k) \right) \gamma_{m} \sum_{j} h(j, j) \times \left( \prod_{k=1}^{r} h(j + \alpha_k, j + \alpha_k) \right) \prod_{k=1}^{m-1} \frac{h(i+j + \beta_k, i+j + \beta_k)}{i+j + \beta_k, i+j + \beta_k} \]  

(9)

From Eqs (8) and (9) we obtain:

\[ P_{r,m} = \sum_{i} h(i, i) \left( \prod_{k=1}^{r} h(i + \tau_k, i + \tau_k) \right) \times C_{m} \left( \alpha_1, \ldots, \alpha_s, i + \beta_{s+1}, \ldots, i + \beta_{m-1} \right) \]  

with \( m \) is pair.

If we suppose \( m = n \) into Eq.(10) we obtain the following equation:

\[ P_{r,2n} = \sum_{i} h(i, i) \left( \prod_{k=1}^{r} h(i + \tau_k, i + \tau_k) \right) \times C_{m} \left( \alpha_1, \ldots, \alpha_s, i + \beta_{s+1}, \ldots, i + \beta_{2m-1} \right) \]

\[ = \sum_{i} h(i, i) \left( \prod_{k=1}^{r} h(i + \tau_k, i + \tau_k) \right) C_{m} \left( \alpha_1, \ldots, \alpha_s, i + \beta_{s+1}, \ldots, i + \beta_{2m-1} \right) \]  

(11)

In the same way, we can write (11) as follows:

\[ P_{r,2n} = \sum_{j} h(j, j) \left( \prod_{k=1}^{r} h(j + \alpha_k, j + \alpha_k) \right) \times C_{n,2} \left( \tau_1, \ldots, \tau_r, j + \beta_1, \ldots, j + \beta_{n-1} \right) \]

(12)

From Eqs(11) and (12) we obtain:

\[ \sum_{i} h(i, i) \left( \prod_{k=1}^{r} h(i + \tau_k, i + \tau_k) \right) C_{m} \left( \alpha_1, \ldots, \alpha_s, i + \beta_{s+1}, \ldots, i + \beta_{2m-1} \right) \]

\[ = \sum_{i} h(j, j) \left( \prod_{k=1}^{r} h(j + \alpha_k, j + \alpha_k) \right) C_{m} \left( \tau_1, \ldots, \tau_r, j + \beta_1, \ldots, j + \beta_{2m-1} \right) \]  

(13)

Based on the Eq (13) we can develop the algorithm that will be described in the following section.

III. PROPOSED ALGORITHM

By setting \( n = 4, \ n - s = r \) and \( r = 2 \), we obtain:

\[ s = 2 \quad \text{and} \quad n - s - 1 = 1 \]

We obtain from Eq(13):

\[ \sum_{i} h(i, i) \left( \prod_{k=1}^{r} h(i + \tau_k, i + \tau_k) \right) C_{4} \left( \alpha_1, \alpha_s, i + \beta_1 \right) \]

\[ = \sum_{i} h(j, j) \left( \prod_{k=1}^{r} h(j + \alpha_k, j + \alpha_k) \right) C_{4} \left( \tau_1, \ldots, \tau_r, j + \beta_1 \right) \]  

(14)

For \( \tau_1 = \tau_2 = q \) and \( \alpha_1 = \alpha_2 = 0 \) the Eq. (14) becomes:

\[ h(0,0) h^2(q,q) C_{4,2} \left( 0,0, \beta_1 \right) = \sum_{j=0}^{q} h^3(j,j) C_{4,2} \left( q, q, j + \beta_1 \right) \]  

(15)

The non linear system is supposed to be causal and has an order \( q \). So, the lag \( (\beta_1) \) must be into the interval \([0,q]\), hence, in order to determine the interval of the variation of the parameter \( \beta_1 \) we precede as follows:

\[ 0 \leq j + \beta_1 \leq q \quad \Rightarrow \quad -j \leq \beta_1 \leq q - j \]  

(16)

and so we have \( 0 \leq j \leq q \). \( \Rightarrow \) \( 0 \leq q - j \leq q \)  

(17)

From Eqs (16) and (17) we obtain:

\[ -q \leq \beta_1 \leq q \]  

(18)

if we consider that \( h(0,0) = 1, h(q,q) \neq 0 \) and the cumulants of order \( m, C_{m,q} (\tau_1, \ldots, t_{m-1}) = 0 \) and if one of the lag \( t_k > q \) and \( t_k < 0 \) with \( k = 1, \ldots, m-1 \), therefore, from equation (16) and (19) we obtain the proposed algorithm characterized by the matrix in this way:
\[
\begin{bmatrix}
0 & \ldots & 0 & C_{q}(q,0) \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & C_{q}(q,q)
\end{bmatrix}
\begin{bmatrix}
\frac{1}{h'(q,q)} \\
\vdots \\
\frac{1}{h'(q,q)}
\end{bmatrix}
= 
\begin{bmatrix}
C_{q}(0,0,q) \\
\vdots \\
C_{q}(0,0,q)
\end{bmatrix}
\]

(19)

Or in more compact form, the Eq. (19) can be written as follows:
\[
A\Theta = b
\]

(20)

With \( A \) the matrix of size \((2q + 1)\times(q + 1)\), \( \Theta \) a column vector of size \((q + 1,1)\) constituted by unknown model parameters and \( b \) is a column of size \((2q + 1,1)\).

The solution based on Least Squares solution (LS) of the system of Eq (18), allows an identification of the non linear model. So, the solution will be written in the following form:
\[
\hat{\Theta} = (A^T A)^{-1} A^T b
\]

(21)

The LS solution gives the estimation of the quotient of the parameters \(\hat{\Theta}(i,i) = \left( \frac{h^3(i,i)}{h^2(q,q)} \right) \), \( i = 1, \ldots, q \). So, in order to estimate the parameters \(\hat{h}(i,i)\), \( i = 1, \ldots, q \) we proceed as follows:

The parameter \(\hat{h}(i,i)\) for \( i = 1, 2, \ldots, q - 1 \) is estimated from the estimated values \(\hat{\Theta}(i,i)\) using the following equation:
\[
\hat{h}(i,i) = \text{sign}[\hat{\Theta}(i,i)]\left(\frac{\hat{\Theta}(q,q)}{\hat{\Theta}(i,i)}\right)^2 \cdot \left(\frac{\text{abs}[\hat{\Theta}(i,i)]}{\text{abs}[\hat{\Theta}(q,q)]}\right)^{\frac{1}{2}}
\]

(22)

Where: \(\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}\), and \(\text{abs}(x) = |x|\) indicates the absolute value of \(x\).

The parameter \(\hat{h}(q,q)\) is estimated by:
\[
\hat{h}(q,q) = \frac{1}{2} \text{sign}[\hat{\Theta}(q,q)] \times \left(\frac{\text{abs}[\hat{\Theta}(q,q)]}{\text{abs}[\hat{\Theta}(q,q)]} + \frac{1}{\hat{\Theta}(1,1)}\right)^{\frac{1}{2}}
\]

(23)

IV. SIMULATION RESULTS

In this section simulations illustrating the performance of the proposed algorithm are provided. The proposed method is applied to non linear finite extent model:

noise free case
\[
y(k) = x^2(k) - 0.35x^2(k - 1) - 0.95x^2(k - 2)
\]

(24)

with noise
\[
S(k) = y(k) + w(k)
\]

(25)

The input sequence \(x(k)\) is white gaussian with variance \(\sigma^2 = 1\). The noise sequence \(w(k)\) is also white gaussian independent of the input. The parameter estimation was performed for different lengths of output sequences and for signal to noise ratio (SNR) levels of \(\approx 0\)dB, 20dB and 0dB.

The SNR is defined as:
\[
\text{SNR} = 10 \log_{10} \left( \frac{E[y^2(k)]}{E[w^2(k)]} \right)
\]

(26)

To measure the accuracy of parameter estimation with respect to the real values, we define the mean square error (MSE) for each run as:
\[
\text{MSE} = \frac{1}{q} \sum_{i=1}^{q} \left( \frac{h(i,i) - \hat{h}(i,i)}{h(i,i)} \right)^2
\]

(27)

Where \(\hat{h}(i,i)\), \( i = 1, \ldots, q \) are the estimated parameters in each run, and \(h(i,i)\), \( i = 1, \ldots, q \) are the real parameters in the model. From the simulation results, presented in Tables I, II and III, we can conclude:

--For all sample sizes and all SNRs, the values of MSES of the proposed algorithm are small, this implies the true parameters are near the estimates parameters.

--The proposed algorithm give a good values the (SD) of estimates parameters, this implies a small variance around the mean value.

In order to test the robustness of the proposed algorithm when the power of noise is very important (SNR=10dB), we consider that \(N=1400\). The (FIGS.1 and 2) demonstrate that our algorithm is not affected by the presence of Gaussian noise because we observe a small fluctuation.

| TABLE I. ESTIMATED PARAMETERS (EQ.24) IN NOISE FREE CASE FOR 50 MONTE CARLO RUNS. TRUE VALUES: h(1,1) = -0.35 and h(2,2) = -0.95 |
|----------------|----------------|----------------|
| N              | \(\hat{h}(1,1)\) ± SD | \(\hat{h}(2,2)\) ± SD | MSE          |
| 400            | -0.3381±0.0244 | -0.9844±0.0108 | 0.0025       |
| 800            | -0.3725±0.0760 | -0.9551±0.0206 | 0.0041       |

| TABLE II. ESTIMATED PARAMETERS (EQ.25) FOR N=1000, WITH DIFFERENT SNR FOR 50 MONTE CARLO RUNS. TRUE VALUES: h(1,1) = -0.35 and h(2,2) = -0.95 |
|----------------|----------------|----------------|
| SNR            | \(\hat{h}(1,1)\) ± SD | \(\hat{h}(2,2)\) ± SD | MSE          |
| 0dB            | -0.3723±0.0758 | -0.9756±0.0500 | 0.0048       |
| 20dB           | -0.3680±0.0623 | -0.9751±0.0342 | 0.0034       |
| 80dB           | -0.3505±0.1376 | -0.9761±0.0456 | 0.0007       |
TABLE III.

ESTIMATED PARAMETERS (Eq.25) FOR SNR=20dB, WITH DIFFERENT N FOR 50 MONTE CARLO RUNS. TRUE VALUES: h(1,1)=-0.35 AND h(2,2)=-0.95

<table>
<thead>
<tr>
<th>N</th>
<th>h(1,1) ± SD</th>
<th>h(2,2) ± SD</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>-0.3539± 0.1379</td>
<td>-0.9649± 0.0722</td>
<td>0.00130</td>
</tr>
<tr>
<td>800</td>
<td>-0.3460± 0.1346</td>
<td>-0.9812± 0.0428</td>
<td>0.00120</td>
</tr>
<tr>
<td>1200</td>
<td>-0.3579± 0.1213</td>
<td>-0.9742± 0.0340</td>
<td>0.00036</td>
</tr>
</tbody>
</table>

V. CONCLUSION

In this paper, we have proposed an algorithm for blind identification of non-linear systems. This algorithm is based on 4th order cumulants. The simulation results demonstrate the fast convergence of the proposed algorithm and its ability to estimate the parameters of the non-linear system.

REFERENCES