A Comparative Study of Coded OFDM Systems

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***Abstract*— The performance of uncoded orthogonal frequency division multiplexing (OFDM) over fading channels is generally improved by introducing some kind of coding. Different coding schemes for OFDM has been reported in the literature. Our aim in this paper is to compare the performance of different coded OFDM systems and compare their bandwidth-complexity tradeoffs. We consider three different coded OFDM systems that employ some kind of non-redundant precoding or redundant postcoding. The comparison of these systems leads us to propose a new hybrid system that employs non-redundant precoding as well as redundant postcoding at same complexity level. Simulation results show that the hybrid system performs better than the other systems considered.**

I. Introduction

Orthogonal frequency-division multiplexing (OFDM) offers several advantages like resilient to multipath fading, intersymbol interference, low complexity and others, and believed to be a promising technique for future broadband wireless communications [1].

While OFDM systems convert a multipath fading channel into a series of equivalent single path parallel channels, they lack the inherent diversity available in multipath channels. Different coded OFDM systems have been reported that employ some form of channel coding or precoding [2], [3] to improve system's performance. The error analysis of communication systems over fading channels shows that it is the Hamming distance (we define it later in the paper) that governs the performance over fading channels [4]. It is important to mention that the Hamming distance of a signal constellation can be increased by non-redundant [5] or redundant coding [3].

Our aim in this paper is to compare the performance of different coded OFDM systems and compare their bandwidth-complexity tradeoffs. We consider three different coded OFDM systems that employ some kind of non-redundant precoding or redundant postcoding. The comparison of these systems leads us to propose a new hybrid system that employs non-redundant precoding as well as redundant postcoding at same complexity level.

The remainder of the paper is organized as follows: Section II presents the system details while Section III discusses different kinds of coded OFDM systems. In Section IV, we discuss the error analysis of coded OFDM systems over fading channels and highlight the importance of Hamming distance that is critical to code design. Section V presents a detailed discussion of the three coded OFDM systems we considered along with the new hybrid system we proposed. We present simulation results in Section VI and conclude the paper in Section VII.

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II. System Details

Consider an uncoded OFDM system that is implemented by using N-point IFFT/FFT. The information symbols are mapped to signal space according to the modulation scheme. The serial stream of modulated data symbols $b(n)$ are grouped in blocks of size N such that the ith block is expressed as $\mathbf{b}(i) := [b(iN),\ b(iN+1)\ \cdots\ b(iN+N-1)]$. Let $\mathbf{F}_N$ is the $N \times N$ FFT matrix with (n,k)th entry as

$$[\mathbf{F}_N]_{n,k} = (1/\sqrt{N}) \exp\{-j2\pi(n-1)(k-1)/N\}. \quad (1)$$

Ignoring the block index i, the output of IFFT block is an OFDM symbol in the form of $N \times 1$ vector and is given by

$$\mathbf{x} = \mathbf{F}_N^{\mathcal{H}} \mathbf{b}. \quad (2)$$

The insertion of the cyclic-prefix (CP) at the transmitter and CP-removal at the receiver, renders the channel matrix $\mathbf{H}$ as an $N \times N$ circulant matrix $\widetilde{\mathbf{H}}$ and the received OFDM symbol can be expressed as:

$$\mathbf{r} = \widetilde{\mathbf{H}}\mathbf{x} + \tilde{\boldsymbol{\eta}} = \widetilde{\mathbf{H}}\mathbf{F}_N^{\mathcal{H}}\mathbf{b} + \tilde{\boldsymbol{\eta}}, \quad (3)$$

where $\tilde{\boldsymbol{\eta}}$ represents the $N \times 1$ additive Gaussian noise vector. At the receiver, the multiplication with FFT matrix $\mathbf{F}_N$ diagonalizes the channel matrix $\widetilde{\mathbf{H}}$ such that it contains the N point discrete frequency response of the channel given by [6]:

$$\mathbf{F}_N \widetilde{\mathbf{H}} \mathbf{F}_N^{\mathcal{H}} = \mathbf{H}_D = \mathrm{diag}\left[\mathbf{F}_\mathrm{N} \tilde{\mathbf{h}}\right], \quad (4)$$

where $\tilde{\mathbf{h}}$ is $N \times 1$ vector obtained from the concatenation of L_h channel taps, $\{h(l)\}_{l=1}^{L_h}$, and $N - L_h$ zeros. Thus, the received OFDM symbols can be simply written as:

$$\mathbf{u} = \mathbf{H}_D \mathbf{b} + \boldsymbol{\eta} \quad (5)$$

The diagonalization of $\widetilde{\mathbf{H}}$ converts an ISI channel into ISI free channel and eliminates the need of complex receiver. Although OFDM systems provide a means to have simple receivers, the system performance deteriorates severely in the presence of channel frequency nulls. This deterioration can be avoided by employing some kind of explicit diversity or redundancy (coding) in the OFDM symbols.

III. Coded OFDM Systems

Depending on the ease of implementation, the coding process can be called before or after the IFFT block in the transmitter as shown in Fig. 1. We term the former as precoded OFDM and in this case the transmitted OFDM symbols can be written as:

$$\mathbf{y} = \mathbf{F}_N^{\mathcal{H}}\underline{\mathbf{A}}\mathbf{b}. \tag{6}$$

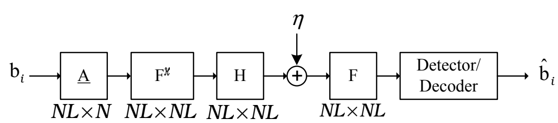
We term the latter as postcoded-OFDM (PC-OFDM) [7] and in this case we encode the OFDM symbols after IFFT as:

$$\mathbf{y} = \mathbf{A}\mathbf{F}_N^{\mathcal{H}}\mathbf{b}. \tag{7}$$

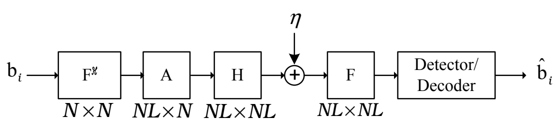
In both cases, we consider complex field coding *i.e.*, $\mathbf{A} \in \mathbb{C}$, instead of Galois field as it provides more degrees of freedom [3]. It is important to note that the two schemes can be made equivalent by selecting

$$\underline{\mathbf{A}} = \mathbf{F}_N\mathbf{A}\mathbf{F}_N^{\mathcal{H}} \tag{8}$$

Another important factor in the design of encoding matrix is the availability of bandwidth. If the system can tolerate a decrease in bandwidth efficiency, it is always desirable for the sake of systems' performance to use redundant encoding where $\mathbf{A}$ (or $\underline{\mathbf{A}}$) has a tall structure of $K \times N$ with $K > N$. Similarly, to save bandwidth one can use non-redundant coding by selecting $\mathbf{A}$ with square structure, *i.e.*, $N \times N$. An obvious advantage of postcoding over precoding is the savings in IFFT module especially for redundant case. Before discussing these possible choices in detail, we first outline the general criterion used to construct "good" encoding matrices in the next section.



(a) Precoded OFDM system



(b) Postcoded OFDM system

Fig. 1. Precoded vs. Postcoded OFDM systems

IV. Code design criterion for fading channels

It has been shown in the recent research that the criteria commonly used to design codes for additive white Gaussian noise (AWGN) channels have to be adjusted when dealing with a fading channel (see [4] and references therein). As we shall see soon, the performance of a code over fading channels depends on the minimum Hamming distance and not on the Euclidean distance between codewords. To see how the choice of encoder affects the system performance, consider the precoding scheme of Fig. 1(a) where the received symbol can be expressed as:

$$\mathbf{u} = \mathbf{H}_D\underline{\mathbf{A}}\mathbf{b} + \boldsymbol{\eta} \tag{9}$$

To assess the system performance over uncorrleted fading channels, we adopt the average pairwise error probability (PEP) technique that has been derived in similar context in [3], [8]. By definition, PEP is the the probability of erroneously detecting $\mathbf{b}'$ while $\mathbf{b}$ was transmitted. We consider ML detection and perfect channel knowledge at the receiver. In order to find PEP (see [3] for details), we need to define a matrix $\mathbf{A}_e := (\mathbf{D}_e\mathbf{V})^{\mathcal{H}}\mathbf{D}_e\mathbf{V}$ where $\mathbf{V}$ is truncated FFT matrix with $[\mathbf{V}]_{(k,l)} = e^{-j2\pi kl/NL}$ and $\mathbf{D}_e = \underline{\mathbf{A}}(\mathbf{b} - \mathbf{b}')$. Now, for Rayleigh fading channels with uncorrelated paths, PEP is given by:

$$\Pr(\mathbf{b} \to \mathbf{b}') \leq \left(\frac{1}{4\mathrm{N_o}}\right)^{-\mathrm{G_d}} \left(\prod_{\mathrm{l}=1}^{\mathrm{G_d}} \alpha_{\mathrm{l}}\lambda_{\mathrm{e,l}}\right)^{-1}, \tag{10}$$

where $N_o/2$ is the power spectral density of additive white Gaussian noise, $\alpha_l = \mathrm{E}[|h(l)|^2]$ is the channel correlation and λ_e are the eigenvalues of $\mathbf{A}_e$. It can be seen from (10) that PEP depends on the following two factors:

- *Diversity gain (G_d):* Roughly speaking, diversity gain represents the slope of the PEP curve especially at high SNR. It is related with the rank of $\mathbf{A}_e$ [8].
- *Coding gain (G_c):* It controls the shift in the PEP curve and depends on the product of eigenvalues $\{\lambda_{e,l}\}_{l=1}^{L_h}$ of $\mathbf{A}_e$ such that $G_c = \left(\prod_{l=1}^{G_d} \lambda_{e,l}\right)^{1/G_d}$

It was shown in [3] that the rank of $\mathbf{A}_e$ is related to the minimum Hamming distance of the codewords. If $\underline{\mathcal{A}}$ is the set of codewords such that $\mathbf{A}, \mathbf{A}' \in \underline{\mathcal{A}}$ then the Hamming distance $\delta(\mathbf{A}, \mathbf{A}')$ between these codewords is the number of non-zero entries in $\underline{\mathbf{A}}(\mathbf{b} - \mathbf{b}')$. The minimum Hamming distance of the codeset $\underline{\mathcal{A}}$ is defined as :

$$\delta_{\min}(\underline{\mathcal{A}}) = \min\{\delta(\mathbf{A}, \mathbf{A}')|\mathbf{A}, \mathbf{A}' \in \underline{\mathcal{A}}\}. \tag{11}$$

The second parameter that controls the shift in the PEP curve is the coding gain. However, it is obvious from (10) that since G_d appears as exponent it can affect the system performance more than G_c.

V. Comparison of Coded OFDM Systems

In this section, we compare three different coded OFDM systems that employ non-redundant precoding or redundant postcoding. The study of these systems leads us to propose a new hybrid system that employs non-redundant precoding as well as redundant postcoding.

A. Non-redundant Precoded OFDM (NR-OFDM)

It is obvious from the discussion in Section III that Hamming distance plays a major role in determining the system performance. While saving the bandwidth, the system performance can still be improved by using non-redundant coding with $\mathbf{A}$ (or $\underline{\mathbf{A}}$) $\in \mathbb{C}^{N\times N}$, *i.e.*, unity code rate. An example of non-redundant coding is *signal space diversity* where the original signal constellation is mapped to a lattice constellation of larger Hamming distance. The name signal space diversity reminds that it is the choice of signal space that increases the diversity. A simpler way to achieve this is by choosing $\underline{\mathbf{A}}$ as a rotation matrix that can rotate the signal constellation to increase the Hamming distance [4]. The design of rotational matrices for signal space diversity is discussed in [5]. Fig. 2 illustrates the application of signal space diversity where the Hamming distance of 4-PSK is increased from 1 to 2 by selecting $\underline{\mathbf{A}}$ as

$$\underline{\mathbf{A}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{j\pi/8} \\ 1 & -e^{j\pi/8} \end{bmatrix}. \tag{12}$$

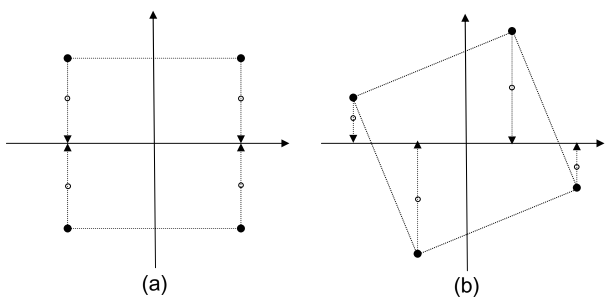


Fig. 2. Effect of signal space diversity on 4-PSK. (a) Without signal space diversity, $\delta_{\min}(\underline{\mathcal{A}})$ = 1. (b) With signal space diversity, $\delta_{\min}(\underline{\mathcal{A}})$ = 2

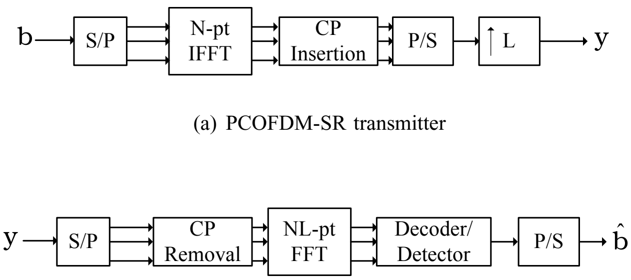
The first coded OFDM system, we consider in this paper, applies signal space diversity to OFDM system by selecting $\underline{\mathbf{A}}$ as rotational matrix. Because of non-redundant coding, we apply this rotational matrix in the form of precoding and term this system as *non-redundant precoded OFDM* or in short 'NR-OFDM'. Due to increased Hamming distance, the application of signal space diversity helps improve the performance of OFDM system without sacrificing the bandwidth efficiency. This assertion will be confirmed through simulations.

B. Postcoded OFDM with simple repetition (PCOFDM-SR)

The second coded OFDM considered in this paper employs redundant coding such that the encoding matrix $\mathbf{A}$ (or $\underline{\mathbf{A}}$) $\in \mathbb{C}^{K \times N}$ with $K > N$. While the recent research emphasizes redundant precoding [9], we explore the use of redundant postcoding in OFDM systems. In postcoding, the redundant encoding is performed after IFFT that leads to a reduced complexity IFFT in the transmitter.

The encoder in postcoded OFDM system introduces explicit frequency diversity in OFDM symbols that can be fairly easily achieved by upsampling the output of IFFT by L [10]. Since upsampling the signal in time domain creates multiple replicas of the signal in frequency domain, this operation is equivalent of repeating the modulated source symbols prior to IFFT. That's why we termed this system as *postcoded OFDM with simple repetition (PCOFDM-SR)*. The block diagram of PCOFDM-SR system is shown in Fig. 3. This particular design of encoder will render the $NL \times N$ postcoding matrix as :

$$\mathbf{A} = \begin{cases} [\mathbf{A}]_{n,k} = \frac{1}{NL} & \text{for } (n,k) = (iL, i) \text{ for } i = 1, \cdots, N \\ 0 & \text{otherwise} \end{cases}$$



(a) PCOFDM-SR transmitter

(b) PCOFDM-SR receiver

Fig. 3. PCOFDM-SR transceiver block diagram

We find that the concept of equivalent precoding matrix of postcoded OFDM facilitates the decoder design. The equivalent precoding matrix for PCOFDM-SR is given by:

$$\underline{\mathbf{A}} = \frac{1}{\sqrt{L}} \begin{bmatrix} \mathbf{I}_N \\ \vdots \\ \mathbf{I}_N \end{bmatrix}, \tag{13}$$

With the equivalent precoding matrix $\underline{\mathbf{A}}$ as defined in (14) and the assumption that the receiver has channel information, the ML decoder is given by

$$\hat{\mathbf{b}} = \min_{\mathbf{b}_i} ||\mathbf{u} - \mathbf{H}_D \underline{\mathbf{A}} \mathbf{b}_i||.$$

ML detection algorithm is computationally extensive but provides the best performance. Other choices of suboptimum detectors include linear detectors like zero forcing and minimum mean square error detectors [3].

C. Postcoded OFDM with linear combination (PCOFDM-LC)

The third coded OFDM system, we consider here, is an extension to PCOFDM-SR that was first introduced in [7]. Since upsampling alone cannot increase the Hamming distance, we therefore multiply the upsampler output with unit magnitude complex number sequence. This is equivalent of generating the linear combination of signals in frequency domain. Thus, we term this system as *postcoded OFDM with linear combination (PCOFDM-LC)*. We showed in [7] that this indeed increases the Hamming distance of the codeset. The block diagram of PCOFDM-LC system is shown in Fig. 4. This particular design of encoder will render the $NL \times N$ postcoding matrix as :

$$\mathbf{A} = \begin{cases} [\mathbf{A}]_{n,k} = \frac{e^{jn}}{NL} & \text{for } (n,k) = (iL, i) \text{ for } i = 1, \cdots, N \\ 0 & \text{otherwise} \end{cases}$$

To gain some insight into PCOFDM-LC, we find the equivalent precoding matrix for this system by using (8).

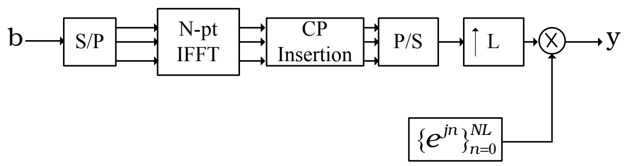


Fig. 4. PC-OFDM-LC transmitter block diagram

Without going into the details (see [7] for details), we discuss this through following example.

Example 1: Consider the design of PCOFDM-LC encoder for $N = 2$ and $L = 2$. The postcoding and equivalent precoding matrices are:

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} e^{j1} & 0 \\ 0 & 0 \\ 0 & e^{j3} \\ 0 & 0 \end{bmatrix}, \underline{\mathbf{A}} = \frac{1}{4}\begin{bmatrix} e^{j1}+e^{j3} & e^{j1}-e^{j3} \\ e^{j1}-e^{j3} & e^{j1}+e^{j3} \\ e^{j1}+e^{j3} & e^{j1}-e^{j3} \\ e^{j1}-e^{j3} & e^{j1}+e^{j3} \end{bmatrix},$$

This bears a close resemblance with rotation matrix of signal space diversity codes (cf. (12)). Thus in a sense, the PC-OFDM-LC system does perform signal constellation rotation through the multiplication with unit amplitude phasors and improves the system performance over fading channels.

D. Nonredundant precoding in PCOFDM-SR (NR-PCOFDM-SR)

The comparison of previous systems suggests that the two important factors in the design of coded OFDM systems that help improve the system performance are: 1) frequency diversity introduced by upsampling, and 2) signal space diversity obtained through constellation rotation. Thus, instead of implicit constellation rotation as in PCOFDM-LC, we propose an explicit constellation rotation in the form of non-redundant precoding to PCOFDM-SR and refer to this system as NR-PCOFDM-SR. Hence, the equivalent precoding matrix for this system is:

$$\underline{\mathbf{A}} = \frac{1}{2}\begin{bmatrix} \mathbf{I}_2 \\ \mathbf{I}_2 \end{bmatrix}\begin{bmatrix} 1 & e^{j\pi/8} \\ 1 & -e^{j\pi/8} \end{bmatrix}, \tag{14}$$

VI. Simulation Results

We perform simulations to compare the bit error rate (BER) of uncoded and coded OFDM systems as shown in Fig. 5. The information symbols are QPSK modulated to yield $\mathcal{B} = \{\pm 1 \pm j\}$. The simulations are performed over Rayleigh fading channel with five taps that are generated according to the Jakes model.

From Fig.5, it can be seen that the use of signal space diversity in NR-OFDM improves the system performance over uncoded OFDM by almost 2dB to achieve the same BER of 10^{-4}. The use of frequency diversity in PCOFDM-SR improves the performance even further and gives an overall advantage of around 2.5dB over uncoded OFDM. The third system, PCOFDM-LC, that introduces frequency diversity and an implicit signal space diversity gives much better results and leads to an improvement of 4.5dB over uncoded OFDM. The new hybrid system we propose in this paper, NR-PCOFDM-SR, gives the best performance though close to PCOFDM-LC.

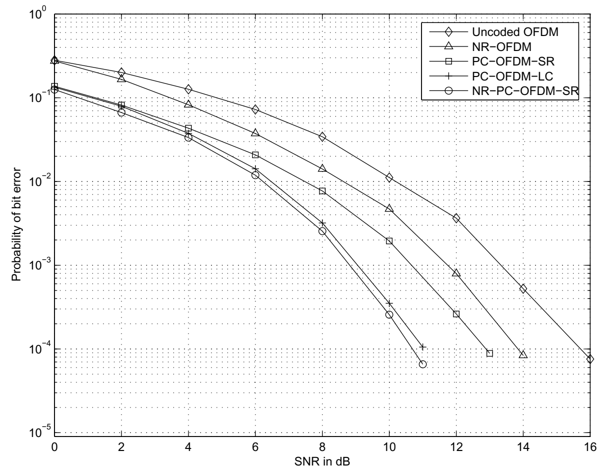


Fig. 5. BER comparison of different coded OFDM systems

VII. Conclusions

We presented a comparative study of different coded OFDM systems and compare their bandwidth-complexity tradeoffs. We consider three different coded OFDM systems that employ some kind of non-redundant precoding or redundant postcoding. The comparison of these systems leads us to propose a new hybrid system that employs non-redundant precoding as well as redundant postcoding at same complexity level. Simulation results show that the hybrid system performs better than the other systems considered.

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