Complexity Constrained Noise-Free CDMA: Optimal Power Distribution and Spectral Efficiency

Rodrigo de Miguel, and Ralf R. Müller, *Senior Member, IEEE*

Abstract—**The issue of multiuser cooperation in a complexity constrained noise-free CDMA channel is addressed. Multiuser cooperation is imperative if conventional demodulation is expected to be error free for non-orthogonal spreading sequences. It is found that for such a constraint the power distribution ensuring the most fruitful cooperation is that which assigns equal power to all users. As a result the asymptotic capacity and spectral efficiency are highest when all users transmit with the same amplitude. It is also shown that for such a complexity constrained channel the asymptotic spectral efficiency is a boundless monotonically increasing function of the channel load.**

I. Introduction

In code division multiple access (CDMA) systems, many users send information simultaneously to a single receiver over the same bandwidth and physical medium. The spreading sequences the different users employ need not be orthogonal, which causes multiuser interference. It is clear that with infinite length random codes or low-density parity check codes, error free communication is possible despite the presence of interference, in theory and in practice, respectively.

In a recent contribution Shental et. al [1] found that even with a constraint on complexity (and delay) communication can be error free in the presence of interference if the number of users is large. The complexity constraint can be very tight and even force the receiver to be a simple standard receive filter matched to signature of the user of interest. Even without the new scheme in [1], such properties of a communication system can be achieved by linear pre-equalization [2] at the receiver if the system is not overloaded. For linear pre-equalization, the transmitter would send continuous amplitude signals instead of binary chips which result from the output of a linear pre-equalization filter. However, Shental et al. [1] showed that the transmitter need not be able to process continuous amplitude data, but may stick to sending binary signals.

Under this constraint, the transmitted signal must be equal to the output signal of the conventional demodulator (which is also known as single-user matched filter [3]) [4]. Shental et al. [1] showed that under such a simple receiving scheme significant asymptotic capacities can be achieved. However, in order for such a scheme to work, cooperation between the sending users is imperative. This calls for an investigation of the power allocation that would allow optimal multiuser coordination. In this work we show that the power distribution allowing for the most fruitful cooperation is that under which all users are assigned equal power. It is also concluded that in such a system the asymptotic spectral efficiency is a boundless monotonically increasing function of the channel load. Spectral efficiency of conventional demodulation without user cooperation is known [5] to also monotonically increase with the load, but to be upper bounded by 1.44 bits/chip.

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Rodrigo de Miguel (corresponding author, email: demiguel@iet.ntnu.no) and Ralf R. Müller (email: ralf@iet.ntnu.no) are with the Department of Electronics and Telecommunications at the Norwegian University of Science and Technology, 7491 Trondheim, Norway.

II. Channel Model

Consider a K-bit input representing K synchronous users. The input bits $i_k = \pm 1$, $k = 1, \cdots, K$ are contained in a column vector $\mathbf{i}$. The k^{th} user transmits with amplitude $a_k \geq 0$, which is the k^{th} eigenvalue of the diagonal matrix $\mathbf{A}$. The spreading sequences of the users are contained in a $N \times K$ rectangular matrix $\mathbf{S}$ with elements $s_{nk} = \pm 1$ ($n = 1, \cdots, N$; $k = 1, \cdots, K$), where N is the spreading factor, and s_{nk} is the n^{th} chip in the spreading sequence of user k.

If the input is sent through a noise-less CDMA channel characterized by the spreading matrix $\mathbf{S}$, then the signal $\mathbf{Ai}$ is modulated onto a N-bit signal $\mathbf{m}$ as

$$\mathbf{m} = N^{-1/2}\mathbf{SAi} \,. \tag{1}$$

By conventional demodulation [4] we may retrieve the original input as a K-dimensional output $\boldsymbol{\o}$ performing the following operation:

$$\boldsymbol{\o} = N^{-1/2}\mathbf{S}^{+}\mathbf{m} = \mathbf{Ai} + \left(N^{-1}\mathbf{S}^{+}\mathbf{S} - \mathbf{I}\right)\mathbf{Ai} \,, \tag{2}$$

where $\mathbf{S}^{+}$ is the Hermitian conjugate of $\mathbf{S}$, and $\mathbf{I}$ is the unit matrix. The first term in the right hand side of (2) is the original signal, and the second term is the multiuser interference. The demodulation concludes by taking the sign of the output

$$\hat{\mathbf{i}} = \operatorname{sgn}\left(\boldsymbol{\o}\right), \tag{3}$$

where the sign is taken component-wise and the hat in $\hat{\mathbf{i}}$ denotes the demodulated input as opposed to the true input $\mathbf{i}$.

Let us now force the input vector $\mathbf{i}$ be such that the recovered signal is equal to the original input

$$\mathbf{i} \equiv \hat{\mathbf{i}} = \operatorname{sgn}\left(\boldsymbol{\o}\right). \tag{4}$$

An input $\mathbf{i}$ for which the constraint (4) holds satisfies the following condition for $\beta = K/N$ [1]:

$$\int_{\left(-\beta^{-1},\infty\right)^K} \mathrm{d}\lambda^K \ \delta\left\{\left(K^{-1}\mathbf{S}^{+}\mathbf{S}-\beta^{-1}\mathbf{A}^{-1}\right)\mathbf{A}\mathbf{i}-\mathbf{\Lambda}\mathbf{i}\right\}=1\,, \tag{5}$$

where the K variables of integration are the K eigenvalues of the diagonal matrix $\mathbf{\Lambda}$.

Given the constraint (4)-(5), the total number $\mathbb{N}$ [1] of valid inputs for K users with amplitude distribution $\mathbf{A}$ and spreading factor N is given as a function of the channel state $\mathbf{S}$:

$$\begin{aligned}&\mathbb{N}\left(\mathbf{S};K,\beta,\mathbf{A}\right)=\\&=\sum_{\mathbf{i}}\int_{\left(-\beta^{-1},\infty\right)^K} \mathrm{d}\lambda^K \ \delta\left\{\left(K^{-1}\mathbf{S}^{+}\mathbf{S}-\beta^{-1}\mathbf{A}^{-1}\right)\mathbf{A}\mathbf{i}-\mathbf{\Lambda}\mathbf{i}\right\}\cdot\end{aligned} \tag{6}$$

III. Channel States and Self-Averaging

A. Channel Microstates and Macrostates

While *microscopically* we specify the state of the channel as one of 2^{NK} equiprobable states $\mathbf{S}$ (henceforth microstates), *macroscopically* the behavior, or macrostate, of the channel may be expressed as the number $\mathbb{N}$ of inputs it allows to fulfill the constraint. Although the specific set of such inputs depends on the channel microstate, it is only the cardinality of the set that determines a macrostate. Therefore, more than one microstate may correspond to the same macrostate.

Let $\Omega\left(\mathbb{N};K,\beta,\mathbf{A}\right)$ be the number of channel microstates that allow exactly and no more than $\mathbb{N}$ inputs to satisfy the constraint, that is the number of microstates that correspond to macrostate $-\mathbb{N}$ for K users, a channel load β, and an amplitude distribution $\mathbf{A}$. There are a total of 2^{NK} possible microstates, and 2^K possible macrostates, which entails

$$\sum_{\mathbb{N}=1}^{2^K}\Omega\left(\mathbb{N};K,\beta,\mathbf{A}\right)=2^{NK}\ . \tag{7}$$

Fig. 1 shows a macrostates-microstates diagram for the case $N = K = 4$ if all the users send with identical amplitude. Because all of the microstates are equiprobable we may directly convert the number of microstates into probability dividing by 2^{NK}. For the case shown in Fig. 1, the most likely macrostate (that is, the one realized by more microstates than any other) is that which allows six of the sixteen possible inputs to fulfill the complexity constraint.

B. Self-Averaging

When the number of users and the spreading factor go to infinity $(K,N\to\infty;\ K/N=\beta)$ one macrostate becomes much more probable than all others. As a result fluctuations away from the most likely macrostate vanish and the

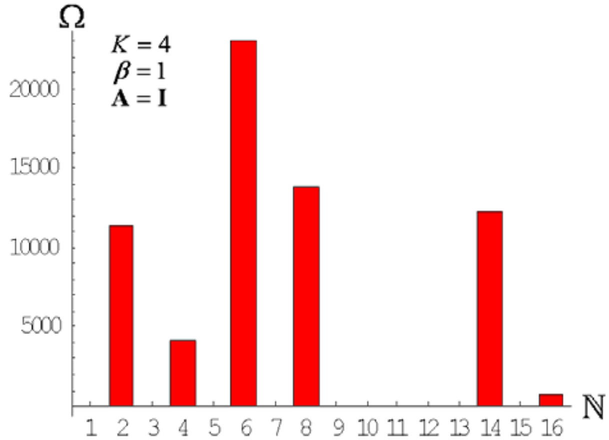

Fig. 1. Macrostates-microstates diagram for a channel consisting of four chips per symbol and four users sending with equal amplitude. The most likely macrostate is that which allows six inputs to fulfill the complexity constraint.

microstate (probability) distribution converges to its own average in what is known as the thermodynamic limit [6]. By virtue of this self-averaging property, we may write

$$\begin{aligned}\mathbb{N}\left(\beta,\mathbf{A}\right)&=\lim_{K\to\infty}\mathbb{N}\left(\mathbf{S};K,\beta,\mathbf{A}\right)\\&=\lim_{K\to\infty}\left\langle\mathbb{N}\left(\mathbf{S};K,\beta,\mathbf{A}\right)\right\rangle_{\mathbf{S}}\end{aligned}, \tag{8}$$

where $\left\langle\xi\right\rangle_{\mathbf{S}}=2^{-NK}\sum_{\mathbf{S}}\xi$ is the configurational average with respect to all channel microstates.

IV. Capacity

As argued, in the infinitely many users limit the number of successful inputs depends only on the channel load and the users' amplitude distribution. The asymptotic capacity of the channel defined in Section II is given, in bits per symbol per user, by the following expression [1]:

$$C_\infty\left(\beta,\mathbf{A}\right)=\lim_{K\to\infty}\frac{\log_2\mathbb{N}\left(\beta,\mathbf{A}\right)}{K}\,. \tag{9}$$

Following a mathematical procedure analogous to that found in [1], the number of successful inputs in the $K\to\infty$ limit is found to be

$$\mathbb{N}\left(\beta,\mathbf{A}\right)=\beta^{-2}K^2 2^{-2}\pi^{-2}\int_{-\infty}^{\infty}\mathrm{d}a\exp\left\{K\ g\left(a;\beta,\mathbf{A}\right)\right\}, \tag{10}$$

where

$$\begin{aligned}&g\left(a;\beta,\mathbf{A}\right)=\frac{1}{2\beta}\times\\&\times\left(\ln\frac{a}{\overline{P}}-1+\frac{\overline{P}}{a}+2\beta\int_0^\infty dPf(P)\ln\left\{\sqrt{\frac{2}{\pi}}\int_{-\sqrt{\frac{P}{\beta a}}}^{\infty}dt\exp\left\{-\frac{t^2}{2}\right\}\right\}\right)\cdot\end{aligned} \tag{11}$$

In (11) $f(P)$ is the asymptotic probability distribution of the users' powers (the eigenvalue distribution of $\mathbf{A}^2$), and $\overline{P}$ is their average power. By virtue of the saddle-point method we may integrate a by making it equal to the value $\tilde{a}$ which maximizes $g\left(a;\beta,\mathbf{A}\right)$; such $\tilde{a}$ must satisfy the following equality:

[1] $\mathbb{N}$ is used as a variable name here and should not be confused with its more frequent meaning, the set of positive integers.

$$\beta\int_0^\infty dPf(P)\left(\int_{-1}^{\infty}dt\exp\left\{-P\left(t^2-1\right)/2\beta\tilde{a}\right\}\right)^{-1}=1-\frac{\overline{P}}{\tilde{a}}. \quad (12)$$

By exploration of (12) it is concluded that, for any given power distribution, $\tilde{a}$ is a function of the load β necessarily greater than or equal to the average power $\overline{P}$ (the equality occurs for vanishing and infinitely large loads). Taking $a=\tilde{a}$ and inserting (10) and (11) into (9) the asymptotic capacity of the channel becomes

$$C_\infty\left(\beta,\mathbf{A}\right)=\frac{1}{2\beta\ln 2}\left(\ln\frac{\tilde{a}}{\overline{P}}-1+\frac{\overline{P}}{\tilde{a}}\right)+ \\ +\frac{1}{\ln 2}\int_0^\infty dPf(P)\ln\left(\sqrt{\frac{2}{\pi}}\int_{-\sqrt{\frac{P}{\beta\tilde{a}}}}^{\infty}dt\exp\left\{-\frac{t^2}{2}\right\}\right). \quad (13)$$

V. Spectral Efficiency

The spectral efficiency is defined as the total capacity per chip, or the total number of bits per chip that can be transmitted reliably. Hence, the asymptotic spectral efficiency η_∞ of the channel is given by the product of the load and the asymptotic capacity:

$$\eta_\infty\left(\beta,\mathbf{A}\right)=\beta\, C_\infty\left(\beta,\mathbf{A}\right). \quad (14)$$

VI. Results and Discussion

A. Asymptotic Capacity

The logarithm multiplying $f(P)$ on the right hand side of (13) is a downward concave function of *P*, and as a result the channel capacity is maximized by a users' power distribution exhibiting zero variance. Such distribution results when all users transmit with identical amplitude, i.e. when $\mathbf{A}$ is proportional to the identity matrix. Fig. 2 shows a plot of the asymptotic capacity of the channel for three power distributions with different variances; for any given channel load the capacity gets larger as the variance decreases.

Although the proposed channel model entails low complexity at the receiver end, it requires strict cooperation among the sending users. For a fraction of users equal to the asymptotic capacity to transmit reliably, all other users must arrange to send bits that will ensure proper decoding at the receiver. This cooperation is best served when all the users send with identical amplitude; otherwise the spreading factor must be increased if the same capacity is to be achieved. The importance of multiuser cooperation is attenuated as the channel load approaches zero and multiuser interference disappears. Hence, the influence of the power distribution disappears for vanishing channel loads. Conversely, as the channel load becomes very large the effect of multiuser interference is overwhelming and any multiuser cooperation is hopeless. It is at moderate loads when the effect of multiuser cooperation in the channel capacity is most significant; see Fig. 2.

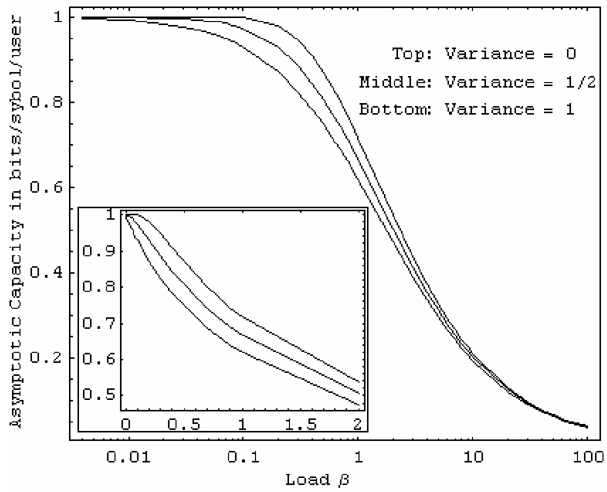

Fig. 2. Asymptotic capacity C_∞ as a function of the load β for different distributions of power among the users. The curves from top to bottom correspond to different chi-square power distributions with average 1 and, respectively, infinite, four, and two degrees of freedom.

B. Spectral Efficiency

As the channel load gets very large its asymptotic capacity may be approximated as follows:

$$\lim_{\beta\to\infty}C_\infty\left(\beta,\mathbf{A}\right)\approx\frac{\beta^{-1/2}}{\ln 2}\int_0^\infty dPf(P)P^{1/2} \quad (15)$$

It is clear from this expansion and Fig. 2 that the spectral efficiency (14) is a monotonically increasing function of the load β and that it decreases with the variance of the power distribution. A plot of the asymptotic spectral efficiency versus the load for different power distributions is shown in Fig. 3.

In order to achieve maximum spectral efficiency the load needs to be infinitely large, which in turn means zero capacity or data transmission rate. Conversely, if the channel were to achieve maximum capacity (which is 1 bit per channel use and user) the bandwidth burnt by spreading would be extremely high, and the spectral efficiency zero. The second law of thermodynamics states that in any real (irreversible) process some power will inevitably be dissipated in the form of entropy. It is only in ideal (reversible) processes that the entropy production vanishes and maximum efficiency can be reached. The only way reversibility may be approached in reality is with an extremely slow energy conversion rate, which deprives the process of any practical significance [7]. Analogously it is only at vanishing data rates that this channel approaches maximum spectral efficiency; and only at the expense of low efficiency that total capacity may be approached. If a balanced trade-off between spectral efficiency and capacity were to be sought, perhaps a convenient channel load would be 1 (as many chips per symbol as there are users), since this is the point where asymptotic capacity and spectral efficiency cross paths as a function of the load as shown in Fig. 4.

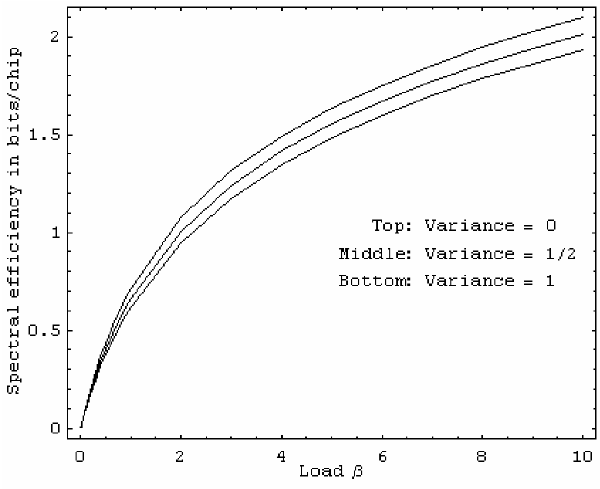

Fig. 3. Asymptotic spectral efficiency η_∞ as a function of the load β for different distributions of power among the users. The curves from top to bottom correspond to different chi-square power distributions with average 1 and, respectively, infinite, four, and two degrees of freedom.

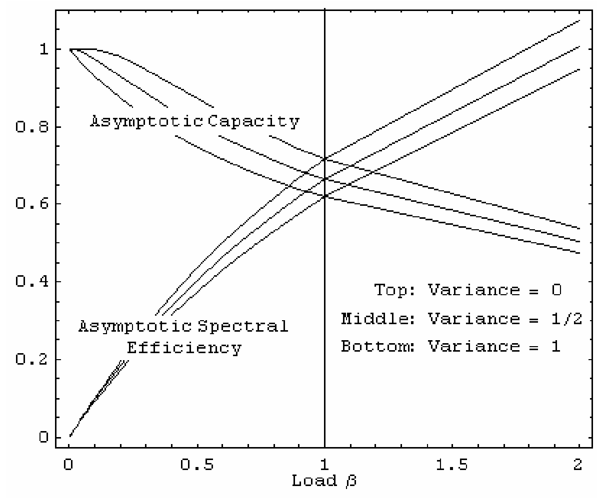

Fig. 4. A plausible tradeoff between asymptotic capacity and spectral efficiency may occur at $\beta = 1$.

VII. CONCLUSION

When the complexity of a CDMA channel is constrained to require only minimal signal processing at the receiver, transmitting users are forced to cooperate. While the receiving task is simplified, the senders must at all times be aware of the channel microstate and coordinate their information accordingly in order to meet the complexity constraint. We have shown that this cooperation is best served when all users transmit with the same amplitude.

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