Super Twisting Algorithm Observer for a Class of Switched Chaotic Systems

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\textbf{Abstract}—The main topic of this paper is the problem of constructing observers for switched systems, which includes, as a specific case, the design of observers based on high order sliding mode technique. The high order sliding mode is used to overcome the occurring chattering phenomena which induces some irrelevant decision of switching between the subsystems when the trajectory is in the neighborhood of the switching manifold. Indeed, a feature of this differentiation structure based on high order sliding modes is the fact that the output does not depend on the discontinuous functions but on an integrator output and that it produce less chattering and avoid the time delays introduced when using a low pass filtering during the computation of the equivalent vector.

\section{I. INTRODUCTION}

Switched systems are class of Hybrid Systems (HS) drawing the attention in the field of control systems theory. Indeed, a large class of system is reasonably modeled by a family of continuous-time subsystems and logic rules that govern the switchings between them. Typically, the switched system consists of several subsystems that switch according to a given switching law and that can be widely found in industries such as process control systems, temperature control systems,.....

Recently, various researchers have studied observability and observer design for such systems. Some sufficient geometrical conditions to analyze the observability of hybrid dynamical systems was given in [6]. These conditions are refined for the particular class of piecewise linear and nonlinear systems. The so-called extended joint observability matrix was proposed in [19], to analyze the observability of jump linear systems. However, the definitions and the testing criteria may vary, depending on the class of systems under consideration and on the knowledge that is assumed on the output. In the same way, other works deal with the hybrid observers design.

Indeed, in [3] a methodology was presented for the design of dynamical observers of hybrid systems that reconstruct the discrete state and the continuous state from the knowledge of the continuous and discrete outputs. A set of observability related definition was introduced in [17]. In [2], the authors focused on the property of the generic final state determinability of HS to design an asymptotic state observer. They showed that this property can be verified even if each of the continuous subsystems of the HS is not observable.

In [7], the design of linear observers for a class of linear hybrid systems was addressed. Two observers prototypes based on the prediction errors were proposed. The first one is based on the observation of an extended discrete-time system. The second one estimates the continuous-time sub-state for all time from initial conditions. Other approaches in the case of state estimation problem for a class of discrete-time piecewise affine systems were presented in [13]. Despite an abundant literature on the design of linear observers for hybrid systems, only few works are concerned with the design of nonlinear hybrid observers for hybrid systems (see for example [14]).

Within this context, we have proposed in [5] a nonlinear observer design for HS without jump. The problem of designing a sliding mode observer for a class of nonlinear hybrid systems was discussed. We have showed that when the trajectory is in the neighborhood of the switching manifold, a chattering phenomena occurs and induces some irrelevant decision of switching between the subsystems. In this work, the main objective is to overcome the drawbacks highlighted in [5] by introducing the so called high order sliding modes (HOSM)[16],[12].

Hence, the main purpose of this paper lies in nonlinear observer design for HS without jump. We discuss the problem of designing a high order sliding mode observer for a class of nonlinear hybrid systems. The considered observers are based on the concept of sliding mode observers and specifically the equivalent control concept [9]. Here, our purpose is to discuss the observer design by using a triangular input observer form introduced in [1], [4] and [9]. The idea consists in using the step by step observer such as described hereafter : The \((n-1)\) first steps consist in reconstructing the state vector and by consequence to know in which discrete state \(P_i\) for \(i = 1, \ldots, k\) the system is found. A complete scheme of the observer is given Figure 1.

The paper is organized as follows: Section II recalls some observability notions for hybrid systems. The high order sliding mode observer design is detailed in section III. In section IV, an illustrative example is presented to show the
II. RECALLS ON OBSERVABILITY STUDY

Observability of nonlinear systems has been extensively and widely studied. More recently, many researchers have approached the study of observability of hybrid systems in general and switched systems in particular. In [19], was considered the case of autonomous switched systems. A definition of observability based on the concept of indistinguishability of continuous initial states and discrete states evolutions from the output in free evolution was given. In [6], both cases of linear and nonlinear switched systems were considered and some algebraic and geometrical conditions of observability for such class were stated.

Since the considered system in this work is a particular class of HS, the system’s observability will be characterized using the observability conditions developed in [6] for such a class of piecewise linear dynamical systems. In what follows is recalled the main result of one of the authors in [6], on the observability of the class of hybrid system considered in this paper. The proof of the theorem can be found in the cited reference. Let us consider the dynamical systems formed with two dynamics interconnected by a switching function:

\[
\begin{align*}
\dot{x} &= f_1(x) \text{ and } y = h_1(x) \text{ if } \sigma(x) \leq 0 \\
\dot{x} &= f_2(x) \text{ and } y = h_2(x) \text{ if } \sigma(x) > 0
\end{align*}
\]

(1)

where \( f_i(x) \) are smooth vector fields, \( h_i(x) \) are smooth outputs and \( \sigma(x) \) is a smooth switching function.

Assumption 1: We assume throughout this paper that:

a) All the evolution duration of each subsystem of (1) are measurable.

b) Each subsystem is observable. That is, for \( i = 1 : 2 \) the codistribution:

\[
\{ dh_i, dL_{f_i}h_i, \ldots, dL_{f_i}^{(n-1)}h_i \}
\]

has rank \( n \)

Assumption 1 a) means that systems with Zeno phenomenon are not considered.

Under assumption 1, if we know which of the subsystem evolves, one can conclude on the observability of the global system (1). Hence, when considering the observability coordinates \( (z^j, j = 1 : 2) \) defined by:

\[
z_{i+1}^j = L_{f_i}^j h_j \quad \text{for } 0 \leq i \leq n - 1
\]

where \( L_{f_i}^j h_j \) is the \( i^{th} \) Lie derivative of \( h_j \) in the direction of \( f_i \), and using the Fliess’s observability canonical form, each subsystem of (1) can be written as:

\[
\begin{align*}
\dot{z}_1^i &= z_{i+1}^1 \\
\ldots
\end{align*}
\]

(2)

if \( \sigma_1 := \phi^{-1}(z_1^1, z_2^1, \ldots, z_n^1) \leq 0 \), and

\[
\begin{align*}
\dot{z}_2^i &= z_{i+1}^2 \\
\ldots
\end{align*}
\]

(3)

if \( \sigma_2 := \phi^{-1}(z_1^2, z_2^2, \ldots, z_n^2) > 0 \).

The approach to analyze the observability of (1), presented in [6], is based on the comparison of \( g_1 \) and \( g_2 \) on the one hand and \( \sigma_1 \) and \( \sigma_2 \) on the other hand. For this, we need to evaluate such functions in terms of the same variables. These variables are given naturally by the output \( y \) and its successive time derivatives \( y^{(i)} = \frac{d^i y}{dt^i} \) for \( i = 1 : n - 1 \).

Let us consider the two submanifolds:

\[
\mathcal{M} = \{ v \in \mathbb{R}^n / g_1(v) = g_2(v) \}
\]

\[
\mathcal{S} = \{ v \in \mathbb{R}^n / \sigma_1(v) = \sigma_2(v) \}
\]

and finally, the submanifold of common singularities of subsystems of system (1):

\[
\mathcal{L} = \{ x \in \mathbb{R}^n / f_1(x) = f_2(x) = 0 \}
\]

The main result is recalled in the following theorem.

**Theorem 1:** [6]

i) If \( \mathcal{M} \) is a discrete set then system (1) is observable for any switch \( \sigma \) for which we have \( \sigma(\mathcal{L}) \leq 0 \) or else \( \sigma(\mathcal{L}) > 0 \).

ii) If dynamics (2) and (3) are transverse to \( \mathcal{M} \) except on a discrete subset then the system is observable for any switch \( \sigma \) for which we have \( \sigma(\mathcal{L}) \leq 0 \) or else \( \sigma(\mathcal{L}) > 0 \).

iii) If \( \mathcal{S} = \mathbb{R}^n \) then system (1) is observable.

The reader can refer to [6] for proof and more details. Some algebraic sufficient conditions on the observability of piecewise linear systems can also be found.

III. HYBRID OBSERVER

In this paper will be designed a step by step sliding mode observer. The idea consists in using the concept of equivalent vector (see [8] and [9]) in an iterative way. Let us consider the canonical observer form of the following nonlinear autonomous system ([10]):
where $y = x_1$. The considered class of systems is assumed to be bounded state in finite time without jumps and does not concern Zeno phenomena.

From the works [1], [4] the following type of sliding mode observer was proposed in [5]:

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= f_i(x) \quad i \in \{1, \ldots, p\} \quad \text{if} \quad \sigma_i(x) \text{ is verified.}
\end{align*}
$$

Fig. 2: Structure of the differentiator of order 2

It can be shown that the error between the system and the observer states vanishes in finite time and that the sliding manifolds are reached one by one. Doing this, a subdynamic of dimension one is obtained and consequently, no peaking phenomena appear. More precisely $E_i = 0$ is equal to zero if there exists $j \in \{1, i-1\}$ such that $\dot{x}_j = \dot{x}_j \neq 0$ (by definition $\dot{x}_j = x_j$), else $E_i = 1$ (see [1] for more details).

In (6), the term $\text{sign}_{eq}$ is the equivalent information injection (by analogy with the well known equivalent control, that is to say the control value required to maintain an ideal sliding motion).

**Remark 1:** The assumption of bounded state must concern the whole system. Indeed, subsystems could be perfectly stable, while the global system can be unstable due to the switching phenomenon.

In [5], a step by step sliding mode observer was mainly used for the following reasons: the finite time convergence and the ability to take naturally into account the variable structure. Nevertheless, some difficulties occur due to the chattering phenomena. It induces some irrelevant decision of switching between the subsystems when the trajectory is in the neighborhood of the switching manifold. This problem was bypassed by using a law pass filter during the computation of the equivalent vector; unfortunately, this solution introduces a delay [5]. In this work, we propose a solution that is relevant for the case of switched hybrid systems. The solution consists in using exact and robust sliding mode differentiator of order 2 (Super Twisting Algorithm) and order 3 (see [15]).

**A. Sliding mode differentiator of order 2 (Super Twisting Algorithm)**

The sliding mode differentiator of order 2 (Super twisting algorithm), represented in figure 2.

The sliding mode differentiator is given in its general form by ([11]):

$$
\begin{align*}
\sum_{obs} u &= \ddot{x}_2 \\
\ddot{x}_2 &= f_q(x, \dot{x}_2) + \alpha_1 \text{sign} (x_1 - \dot{x}_1)
\end{align*}
$$

(7)

where $e_1 = x_1 - \dot{x}_1$ and $\lambda_1$ and $\alpha_1$ are positive tuning parameters of the differentiator whose the output is $u_1$ and

$$
sign(e_1) = \begin{cases} +1 & \text{if } e_1 > 0 \\
-1 & \text{if } e_1 < 0 \\
\in [-1, 1] & \text{if } e_1 = 0
\end{cases}
$$

Applying the exact differentiator to system (4) when $n = 2$, one obtains:

$$
\begin{align*}
\dot{x}_1 &= \dot{x}_2 + \lambda_1 |x_1 - \dot{x}_1|^{1/2} \text{sign}(x_1 - \dot{x}_1) \\
\dot{x}_2 &= f_q(x_1, \dot{x}_2) + \alpha_1 \text{sign}(x_1 - \dot{x}_1)
\end{align*}
$$

(8)

with

and the error dynamics is given by:

$$
\begin{align*}
\dot{e}_1 &= e_2 - \lambda_1 |e_1|^{1/2} \text{sign}(e_1) \\
\dot{e}_2 &= f_q(x) - f_q(x_1, \dot{x}_2) - \alpha_1 \text{sign}(e_1)
\end{align*}
$$

(9)

The convergence of the observation error is obtained in one step in finite time. Another feature of the differentiators (7) is the fact that the output does not depend directly on discontinuous functions but on an integrator output. So high frequency chattering is attenuated [5]. Both properties are important since the switching function can be obtained in a continuous way and without delays.

**B. Convergence analysis**

**Theorem 2:** Consider the system (1), assumed to be “bounded state” in finite time, and the observer based on the differentiator (8). For any initial conditions $x(0), \dot{x}(0)$, there exists a choice of $\lambda$ and $\alpha$ such that the observer state $\dot{x}$ converges in finite time to $x$, i.e. $(\ddot{x}, \dot{x}) \to (x_1, x_2)$, and $\sigma(\ddot{x})$ converges to $\sigma(x)$, thus, $\ddot{q} \to q$ and by consequence both the continuous state and the discrete location are known.

**Proof:** The proof is given in [18]. Figures 3 and 4 illustrate the finite time convergence behavior of the proposed observer for switching systems. The demonstration is based on the error trajectory for each quadrant in the worst cases.
defining in which state (location) \( q \), the system is found. Another feature of the differentiator (7) is the fact that the output does not depend directly on discontinuous functions but on an integrator output. So high frequency chattering, which can be very harmful for the chaotic system known for its extreme sensitivity to noise, is attenuated [5]. Both properties are important since the switching function can be obtained in a continuous way and without delays.

The structure of the step by step differentiator for a third order system is given in figure 5.

Proposition 1: Consider the system (4), assumed to be bounded state in finite time, and the observer based on the Super Twisting Algorithm. For any initial conditions \( x(0) \), \( \hat{x}(0) \), there exists a choice of \( \lambda_i \) such that the observer state \( \hat{x} \) converges in finite time to \( x \), and \( \sigma( \hat{x} ) \) converges to \( \sigma(x) \).

Proof: The proof is stated on the basis of the one developed by the authors in [18]. The convergence is ensured step by step following this order: \( (\hat{e}_1 = e_2, e_1) \rightarrow (0, 0) \) in finite time \( T_1 \) in the first step, \( (\hat{e}_2 = e_3, e_2) \rightarrow (0, 0) \) in finite time \( T_2 \) in the second step, and \( (\hat{e}_i = e_{i+1}, e_i) \rightarrow (0, 0) \) in finite time \( T_i \) in the step \( i \), and finally, \( (\hat{e}_{n-1} = e_n, e_{n-1}) \rightarrow (0, 0) \) in finite time \( T_{n-1} \) in the step \( (n-1) \).

The finite time of convergence is given by \( T = \sum_{j=1}^{n-1} T_j \).

D. Discrete time observer

Let us consider the system (4), the task of the discrete time observer is to locate which dynamic of the system is in evolution?. In some cases, the knowledge of the system’s output is sufficient to estimate the current location. If this is not the case, some additional information obtained by using the continuous observer, may be useful or are necessary to estimate the current location.

In our case, the discrete-observer receives as input: the observed state \( \hat{x} \), the output \( y \). Its task is to provide an estimation \( \hat{q} \) of the discrete location \( q \) of the hybrid plant at the current time. This information is then used by the controller to design the adequate control for the dynamics.
$f_q$ in evolution and then to realize the control objective. Contrarily to the general case; here, the continuous observer doesn’t need to know the discrete location $q$. Indeed, the second order sliding mode observer for the second order system has to know only the output $y = x_1$ and also the $f_q(x) - f_q(\dot{x})$ upper bound, noted $g^+$. Thus, one can announce the following result:

Proposition 2: If the observer is sufficiently fast, then $\sigma(\dot{x})$ converges towards $\sigma(x)$. The knowledge of the state estimation and the use of the location identification Logic, lead to conclude to the estimation of $\check{q}$.

Proof: (Sketch of proof) As the system is given in the Flieess observability form, the use of a sliding mode observer allows us to reconstruct the state vector by using only $y = x_1$, and it is not necessary to know the discrete location $q$.

Now to estimate $q$, we use the estimation of $x$ and then evaluate $\sigma$ and finally by using the logic identification, we can give an estimation $\check{q}$ of $q$.

IV. ILLUSTRATIVE EXAMPLE

Let us consider the following switched systems:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \begin{cases} P_1(x) & \text{if } \sigma(x) \geq 1 \\
P_2(x) & \text{if } |\sigma(x)| < 1 \\
P_3(x) & \text{if } \sigma(x) \leq -1 \\
\end{cases}
\end{align*}
\]

with: $P_1(x) = -\frac{1800}{49}x_1 - \frac{55}{7}x_2 - \frac{25}{7}x_3 + \frac{2700}{49}$; $P_2(x) = \frac{100}{49}x_1 - \frac{55}{7}x_2 - \frac{25}{7}x_3$; and $P_3(x) = -\frac{1800}{49}x_1 - \frac{55}{7}x_2 - \frac{25}{7}x_3 + \frac{2700}{49}$. and the switching condition: $\sigma(x) = -\frac{7}{100}(x_3 + x_2) - x_1$

The chaotic behavior of the considered system is represented by its phase portrait in figure 7, this figure shows a trajectory simulation $x$ of the switched nonlinear system, together with the estimated states, while figure 6 highlights the efficiency of the proposed observer and shows the finite time step by step convergence.

Figure 8 shows respectively the switching function $\sigma(x)$ and $\sigma(\dot{x})$ and the corresponding switching indicator $S$, and $S_o$. There is no problem of delay between $S$ (switching indicator calculated on the basis of the real states $x_1$, $x_2$, and $x_3$) and $S_o$ (switching indicator calculated on the basis of the observed states $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$). Also, it can be noted that when $\sigma(\dot{x})$ is near $-1$ around $t = 5s$, there is no undesirable chattering phenomenon and no irrelevant decision of switching between the subsystems when the trajectory is in the neighborhood of the switching manifold as it was the case when classical sliding mode was used with a filter (see figure 9 and paper [5]).

V. CONCLUSION

In this paper, an approach to state observation of switched nonlinear systems was proposed, resulting in a switched observer. A sliding mode differentiators sliding mode observers was carried out. This observer was employed to generate the time derivatives of the output to observe a hybrid system.
The simulation results obtained in the synthesis of a switched observer have been reported. We have shown that these tasks can be accomplished by means of exact differentiation via sliding mode technique which are confirmed again to be a powerful and efficient tool to solve difficult design problems of observation.

REFERENCES