Digital Sound Synthesis by Block-Based Physical Modeling

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Abstract—Different methods have been proposed for digital sound synthesis by physical modeling. Since musical instruments may consist of several components with quite different physical nature, there is no single best modeling paradigm. Choosing the optimal model for each component requires a common interconnection strategy, which leads to a computable algorithm with faithful sound reproduction. This contribution describes such an interconnection strategy on the basis of the wave digital principle. Examples are given for string, membrane, and brass instruments.

I. INTRODUCTION

Digital sound synthesis by physical modeling attempts to model the physical sound production process itself rather than only to recreate a certain waveform. A number of different techniques have been developed so far. However, since many musical instruments consist of different components with different physical nature, there is no single best method for physical modeling. Choosing the optimal model for each component requires a common interconnection strategy, which leads to a meaningful discrete-time approximation of the total system.

Such a strategy is provided by the well-known wave digital principle. The use of wave variables for the interconnection network avoids the formation of delay free loops and yields computable algorithms. This contribution describes the wave digital principle as a starting point for the more general idea of block-based physical modeling. It has been developed as a joint effort by the consortium of the European project ALMA [1]. Section II presents the classical wave digital principle for lumped and distributed parameter networks. Section III introduces block-based modeling with examples for string and membrane instruments in Section III-B, and brass instruments in Section III-C.

II. THE WAVE DIGITAL PRINCIPLE

The wave digital principle was introduced as a method for designing digital filters (wave digital filters) from analog counterparts. A unifying treatment of theory and application of wave digital filters is given in a classical paper by A. Fettweis [2]. Modern descriptions of the wave digital principle as a tool for numerical integration and modeling are given in [3], [4], [5], [6]. The following subsections present a short introduction to the basic idea of the wave digital principle and discuss its properties.

A. A Short Introduction to 1D Wave Digital Filters

This section gives a quick overview on the basic idea of wave digital filters. Consider an energy storage element in a physical model, e.g. a capacitor in an electrical circuit. It relates the voltage $u$ and the current $i$ by integration

$$u(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau) d\tau . \quad (1)$$

$u$ and $i$ are also called Kirchhoff variables. Numerical integration by the trapezoidal rule with the time step size $T$ performs the time discretization according to the bilinear transformation ($k$ is the discrete time index)

$$u((k+1)T) = u(kT) + \frac{T}{2C} [i((k+1)T) + i(kT)] . \quad (2)$$

However, this difference equation is not computable, since neither $u((k+1)T)$ nor $i((k+1)T)$ are known at the time instant $(k+1)T$. To avoid the resulting delay free loop the unknown quantities are sorted according to time

$$u((k+1)T) - \frac{T}{2C} i((k+1)T) = u(kT) + \frac{T}{2C} i(kT) . \quad (3)$$

Then the so-called discrete wave variables are introduced

$$a[k] = u(kT) + R_p i(kT) , \quad (4)$$

$$b[k] = u(kT) - R_p i(kT) . \quad (5)$$

In accordance with physical intuition, $a[k]$ is called the incident wave and $b[k]$ the reflected wave and $R_p$ is the reference resistance.

The introduction of the wave variables is the key element of the wave digital principle. While the across and through variables $u$ and $i$ mutually depend on each other, there is a causal relation between the wave variables $a$ and $b$, which ensures the computability of the resulting structure. This can be seen immediately by expressing (3) in terms of the wave quantities with the reference resistance $R_p = T/2C$. The result of this transition is simply

$$b[k] = a[k-1] . \quad (6)$$

Thus, the numerical integration (3) is computable by expressing the reflected wave $b[k]$ at the time instant $k$ by the incoming wave $a[k-1]$ at the previous time $k-1$.

Table I lists some selected elements of electrical networks and their wave digital counterparts. The behavior of the wave digital equivalents follows from similar derivations as for the capacitor in eqn. (1) to (6). For all linear elements, the reflected wave $b[k]$ is given by known quantities or by the incident wave $a[k-1]$ from the previous time step. For nonlinear elements, the relation between $a$ and $b$ depends on the type of nonlinearity.
Once the wave digital filter equivalents of the single network elements are known, they have to be connected to form the discrete-time equivalent of the whole network. Since the reference resistances for each wave digital element are different (see Table I), the interconnection network has to match the correct reference resistances at each port.

Any interconnection network can be decomposed into two kinds of blocks, a three-port serial and a three-port parallel adaptor [7], shown in Fig. 1. The ports of these adaptors connect to other adaptors to form a larger interconnection network. The internal structure of each adaptor is established by relating the voltages and currents and the corresponding network. The interconnection by adaptors poses a restriction on the blocks to be connected. Since the interconnection is based on wave variables, all blocks need to communicate via wave variables. This demand is automatically fulfilled for wave digital equivalents of networks elements. However many physical models which are derived directly from ordinary or partial differential equations use Kirchhoff variables instead of wave variables. Their discrete-time models are only suitable for an interconnection network if they can be augmented with an interface that converts its inputs and outputs to wave variables. The interconnection of wave digital structures has been developed [7], [8]. Recently, also design methodologies and computer programs for the automated generation of wave digital filters and their extension to multidimensional systems see e.g. [2], [3], [4], [5], [6].

### Table I

**Kirchhoff network elements and their wave digital counterparts. From top to bottom: Resistor, inductance, capacitance, voltage source with internal resistance, nonlinear element.**

<table>
<thead>
<tr>
<th>Network Element</th>
<th>Behavior</th>
<th>Symbol</th>
<th>Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>$u(t) = R \cdot i(t)$</td>
<td><img src="image1" alt="Symbol for R" /></td>
<td>$b[k] = 0$</td>
</tr>
<tr>
<td>$L$</td>
<td>$u(t) = L \cdot \frac{\partial}{\partial t}i(t)$</td>
<td><img src="image2" alt="Symbol for L" /></td>
<td>$b[k] = -a[k-1]$</td>
</tr>
<tr>
<td>$C$</td>
<td>$i(t) = C \cdot \frac{\partial}{\partial t}u(t)$</td>
<td><img src="image3" alt="Symbol for C" /></td>
<td>$b[k] = a[k-1]$</td>
</tr>
<tr>
<td>$u_0$</td>
<td>$u(t) = u_0(t) + R \cdot i(t)$</td>
<td><img src="image4" alt="Symbol for u0" /></td>
<td>$b[k] = u_0[k]$</td>
</tr>
<tr>
<td>NL</td>
<td>nonlinear behavior</td>
<td><img src="image5" alt="Symbol for nonlinear" /></td>
<td>$b[k] = f(a[k])$</td>
</tr>
</tbody>
</table>

### B. Properties of Wave Digital Filters

The basic elements of the wave digital approach have been presented above. For an in-depth coverage of wave digital filters and their extension to multidimensional systems see e.g. [2], [3], [4], [5], [6]. Recently, also design methodologies and computer programs for the automated generation of wave digital structures have been developed [7], [8]. However, in its present form the wave digital principle is an all-or-nothing strategy, since the above design methods require to design the complete system from scratch. Although the wave digital principle allows great flexibility within its framework, there are no provisions to consider models or simulation methods from outside the wave digital world. The next section shows how to overcome this restriction by attaching also physical models according to other paradigms to a wave digital adaptor network.

### III. BLOCK-BASED PHYSICAL MODELING

#### A. Extension of the Wave Digital Principle

The interconnection by adaptors poses a restriction on the blocks to be connected. Since the interconnection is based on wave variables, all blocks need to communicate via wave variables. This demand is automatically fulfilled for wave digital equivalents of networks elements. However many physical models which are derived directly from ordinary or partial differential equations use Kirchhoff variables instead of wave variables. Their discrete-time models are only suitable for an interconnection network if they can be augmented with an interface that converts its inputs and

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Fig. 1. Three port parallel adaptor (top) and three port serial adaptor (bottom) and equivalent networks with Kirchhoff variables.
outputs to wave variables and vice versa. Such an interface is called a Kirchhoff-Wave convertor or K-W-converter and has been described in [9]. Similar to an adaptor which connects wave digital blocks with different port resistances, a K-W-converter connects an arbitrary physical model to an adaptor. For the interconnections between finite difference structures and digital waveguides, an appropriate interface has also been discussed in [10].

The extension of the wave digital principle described above is called block-based physical modeling. It denotes a modeling strategy where different physical modeling methods can be mixed as long as they communicate via wave variables in a compatible interconnection network.

B. String and Membrane Instruments

The physical model of vibrating structures is derived from basic laws of physics [11] and relates the input force \( f_e(x, t) = \gamma_0(x - x_e)f_e(t) \) at the excitation position \( x_e \) (\( \gamma_0(x) \) denotes the spatial delta-impulse) with the deflection \( y(x, t) \) of the vibrating body by

\[
\rho A \dddot{y} - T S \dddot{y}'' + EI_b y'''' + d_1 y' - d_3 \ddot{y}'' = f_e . \tag{7}
\]

where \( \ddot{y} \) denotes temporal derivative and \( y' \) denotes spatial derivative of the string’s deflection.

The model includes the lossless wave equation, with the string tension \( T_S \), the mass density \( \rho \), and the cross section area \( A \). Two additional damping terms with the coefficients \( d_1 \) and \( d_3 \) model frequency independent and frequency dependent damping. The fourth order spatial derivative, which is scaled by Young’s modulus \( E \) and the moment of inertia \( I_b \), models the stiffness of the string.

Frequently used physical models for strings, membranes, plates, and alike are digital waveguides [5], [12] or functional transformation method (FTM) models [13]. Digital waveguides are close relatives of wave digital filters and use also wave variables [4]. FTM models are a direct transformation of the underlying partial differential equation (PDE) into the frequency domain and use Kirchhoff variables. At first sight, they seem to be largely incompatible with the wave digital principle. It is therefore of special interest to investigate how the block-based modeling methods presented before allow to use an FTM model as building block in a larger structure.

The functional transformation method converts the PDE together with suitable initial and boundary conditions into a discrete-time algorithm [13], [14], [15] shown in Fig. 2. The vibration of the body is modeled by the parallel arrangement of first order systems with complex frequencies \( \beta_n, n = 1, \ldots, N \). The weighting constants \( b_n \) and \( c_n \) follow from the application of the functional transformation method. Each first order system is driven by samples of the excitation \( f_e[k] \) and delivers a contribution to the velocity of the vibration \( \dot{y} \). However for the communication with other blocks, the Kirchhoff variables force and velocity are converted to the wave variables \( a[k] \) and \( b[k] \). Thus the discrete-time model which results from the functional transformation method can be connected to wave digital equivalents [16].

Fig. 2. Structure of the string implementation which uses the wave variables \( a[k] \) and \( b[k] \) for input and output instead of the Kirchhoff variables \( f_e[k] \) and \( \dot{y}[k] \).

Fig. 4 shows a simple model of a string excited by picking or plucking. The block on the bottom contains the structure from Fig. 2. The block on the right is an energy source, injecting energy to the string with every note played. The block on the left can be used to model a nonlinearity. These three blocks are connected via their wave variables by a three-port series adaptor from Fig. 1.

Fig. 4. Excitation of the string with a source model to simulate picking.

Fig. 5 shows a membrane with hammer excitation. The excitation mechanism is a felt covered hammer like in a piano. The inductor and the resistor model the dynamics of the hammer motion. The nonlinear block on the left contains the pressure-dependent stiffness of the hammer felt. All these blocks are connected by an interconnection network consisting of a series and a parallel three-port adaptor.

Fig. 5. Hammer membrane model, where the hammer mass corresponds to the inductance and the hammer-felt is modeled by the nonlinearity.

The models presented here have been created with a program for the automated synthesis of wave digital structures. The program has been developed at the Politecnico di Milano [7] in the course of the European project ALMA [1]. It is based on the topology of the binary connection tree and allows one nonlinear block in each model.
C. Brass Instruments

The ability to model complex structures is now highlighted by a model of a brass instrument. Fig. 6 shows the profile of a brass instrument with its four essential parts. The player’s lips are the excitation mechanism and require a nonlinear model. The mouthpiece is a resonator with a bandpass characteristic. The cylindrical air-column acts as an acoustic waveguide and the horn matches the acoustic impedance to the free field.

A block-based physical model is shown in Fig. 3. Appropriate models for each block are described e.g. in [11], [17], [18]. The lips are represented by a nonlinear lumped parameter model. The resonating properties of the mouthpiece are modeled by the wave digital equivalence of a corresponding electrical circuit. The air-column is modeled as a digital waveguide [5]. For the horn, the functional transformation method from Fig. 2 is used. The horn is terminated by a simple resistor to simulate losses due to the sound radiation.

IV. Conclusions

Block-based physical modeling allows to build digital sound synthesis algorithms with a great variety of models, including linear and nonlinear lumped and distributed parameter systems. An interconnection strategy based on the wave digital principle guarantees the computability of the resulting algorithms. They can be generated by an automated synthesis strategy based on the binary connection tree. If appropriate models are chosen for the individual blocks, then the generated algorithms allow real-time processing with low latency.

REFERENCES


