Design of LDPC-Coded Modulations for Fast Iterative Decoding

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Abstract— **In this paper, we propose an EXIT chart-based approach to code design for low-density parity-check (LDPC)-coded modulations. In particular, the adopted technique iteratively modifies the LDPC code degree distribution in order to achieve a target bit error rate (BER) at a given (and possibly small) number of iterations. In this paper, we limit ourselves to communication systems on a binary erasure channel (BEC), but the proposed approach is general. The optimized codes for a BEC give significant insights into the impact of the degree distributions on the BER performance.**

I. Introduction

In the last decade, the flourishing of powerful coding techniques such as turbo codes, the rediscovery of low-density parity-check (LDPC) codes and, in general, a deeper understanding of iterative detection/decoding techniques have allowed to make communication systems perform in the proximity of the channel capacity [1], [2]. Nevertheless, iterative techniques pose a trade-off between computational complexity and system performance. In general, achieving a performance close to the channel capacity calls for high computational complexity. It is, therefore, interesting to investigate coding techniques which, while outperforming traditional techniques such as convolutional and algebraic coding, require similar or even lower computational complexities. Iterative techniques using a small number of iterations seem an appealing solution to this problem.

In [3], a powerful analysis technique is introduced for iterative detection/decoding schemes based on the concept of mutual information (MI) [4] and MI exchange between processing blocks. The basic graphs used in this technique are denoted as extrinsic information transfer (EXIT) charts. In [5], an LDPC code optimization algorithm is proposed for a binary erasure channel (BEC). The codes are optimized in order to minimize the number of iterations needed to achieve a given bit error rate (BER).

In this paper, we investigate LDPC codes for a BEC using EXIT charts. The evolution of the MI in the EXIT charts is used to accurately predict the BER. We fix the number of decoding iterations and optimize the descriptive parameters of the LDPC code, i.e., the degree distributions [6], in order to achieve a given target BER with the "worst possible channel." New insights on the code structure are obtained. In particular, if a small number of iterations (i.e., between 4 and 8) and a low target BER (i.e., lower than 10^{-3}) are considered, the best codes are very similar to *regular* LDPC codes whose variable node degree is a monotonic function of the target BER. In this paper, the optimization is carried out for a simple BEC, the proposed LDPC code optimization framework can be straightforwardly applied to several transmission schemes with arbitrary modulations and channels, both with and without memory.

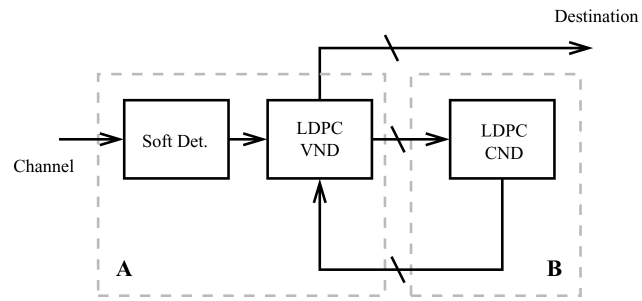

Fig. 1. Schematic representation of the receiver.

The paper structure is as follows. In Section II, EXIT charts are introduced along with their use to predict the BER. In Section III, the asymptotic BER is investigated as a function of the code structure and the number of decoding iterations. In Section IV, the proposed code optimization algorithm is introduced. In Section V, several code optimizations are performed and the generated codes are investigated. Section VI concludes the paper.

II. EXIT Chart-based Analysis and BER Bounds

In Fig. 1, the structure of a belief propagation iterative LDPC decoder is depicted [7]. Two main blocks, exchanging vector reliabilities, can be identified: the block labeled **A** comprises a soft detector for the channel output and the variable node detector (VND), associated with the LDPC code variable nodes; the block labeled **B** comprises the check node detector (CND) associated with the set of LDPC code check nodes.

EXIT charts are a tool which allows approximate (still accurate) analysis of the convergence behavior of iterative detection schemes [3]. In particular, the MI between the transmitted codeword bits and the exchanged reliabilities is used as a measure of the overall achieved system reliability. The EXIT chart-based analysis assumes that the MI at the output of each block in the detector's scheme is a single-valued function of the MI of the reliabilities at the input of the block.

The achieved BER after a given number of iterations can be bounded as follows:

$$H^{-1}[1 - I(X;Y)] \leq \text{BER} \leq \frac{1 - I(X;Y)}{2} \qquad (1)$$

where $I(X;Y)$ is the achieved mutual information between the generic information bit X and the associated reliability Y, the lower bound is the Fano bound applied to a binary random variable (RV), $H(p) \triangleq -p\log(p)-(1-p)\log(1-p)$ is invertible if $p \in (0, 1/2]$ and the upper bound can be found in [8]. The lower bound is independent of the decision strategy and the reliability values distribution. The upper bound is obtained assuming a maximum *a posteriori* (MAP) decision strategy. Both bounds are obtained assuming equiprobable codeword bits and are both tight, in the sense that there exist distributions achieving them.

The bounds in (1) are useful since they allow an accurate estimate of the BER obtained by a system when the MI between a generic codeword bit and the set of the reliability messages associated with this bit is close to 1. In the following, we will use the upper bound in (1) as an estimate of the BER after a given number of LDPC decoder iterations. The BER will be used as a functional which has to be minimized with respect to the descriptive parameters of the LDPC code, i.e., the degree distributions [6].

III. Asymptotic BER for the Binary Erasure Channel

We now characterize the relationship between the number of iterations and the BER, when the iterative LDPC decoder is in the convergence region, i.e., in the last few iterations needed to achieve the desired BER. We focus on a BEC due to its simplicity. Nevertheless, we remark that the characterization relative to a BEC is useful for other channels as well, since it is known that when the decoding process converges at low BER, the system behavior approaches that of an iterative decoder operating on a BEC [9].

The EXIT functions of the VND and CND are given by

$$I_{\mathrm{v}}(I) = 1 - \sum_i \lambda_i (1 - I_{\mathrm{ch}})(1-I)^{i-1} \tag{2}$$

where I_{ch} is the BEC channel capacity, and

$$I_{\mathrm{c}}(I) = \sum_j \rho_j I^{j-1} \tag{3}$$

respectively [10], [11]. The decoding process converges if the recursion

$$\begin{array}{rcll} I_0 & = & I_{\mathrm{ch}} & \\ I_n & = & I_{\mathrm{v}}(I_{\mathrm{c}}(I_{n-1})) & \quad n \geq 1 \end{array} \tag{4}$$

tends to 1 as n tends to infinity. Approximating (3) with its first-order Taylor series expansion centered in 1, one obtains

$$I_{\mathrm{c}}(I) \simeq 1 - (1-I)\sum_j \rho_j (j-1)\,. \tag{5}$$

One can also approximate $I_{\mathrm{v}}(I)$ with a Taylor approximation centered in 1, choosing the minimum order such that the resulting function is not constant:

$$I_{\mathrm{v}}(I) \simeq 1 - \lambda_{i_{\min}}(1-I_{\mathrm{ch}})(1-I)^{i_{\min}-1} \tag{6}$$

where $i_{\min}$ is the lowest variable node degree.

Suppose now that after n_0 iterations the MI is sufficiently close to 1 such that (5) and (6) are good approximations. After n further iterations, the MI is given by the following relation, obtained by substituting (6) and (5) into (4) and writing explicitly the recursion:

$$\begin{aligned} I_{n+n_0} \simeq\ & 1 - \left[(1-I_{\mathrm{ch.}})\lambda_{i_{\min}}\right]^{\frac{b^n-1}{b-1}} \\ & \cdot \left[\textstyle\sum_j \rho_j (j-1)\right]^{b\frac{b^n-1}{b-1}} (1-I_{n_0})^{b^n} \end{aligned} \tag{7}$$

where $b = i_{\min} - 1$. Applying the upper bound on the BER in (1) with $I(X;Y) = I_{n+n_0}$, one obtains

$$\begin{aligned} \mathrm{BER}_{n+n_0} \lesssim\ & \tfrac{1}{2}\left[(1-I_{\mathrm{ch.}})\lambda_{i_{\min}}\right]^{\frac{b^n-1}{b-1}} \\ & \cdot \left[\textstyle\sum_j \rho_j (j-1)\right]^{b\frac{b^n-1}{b-1}} (1-I_{n_0})^{b^n}\,. \end{aligned} \tag{8}$$

Considering (8), one can observe that the behavior of the BER as a function of the number of iterations in the convergence region is *exponential of exponential*. In other words, the BER should exhibit a "threshold behavior," both as a function of iteration number n and as a function of the minimum variable node degree $i_{\min}$. This result is useful if the goal is to design LDPC codes which achieve very low BER after a small number of iterations, i.e., codes which (i) can be decoded with low complexity, (ii) are characterized by low decoding delay and (iii) do not need the presence of an algebraic concatenated coding scheme in order to guarantee "error-free" performance. This also shows that, in order to reach very low BER, a code with low-degree variable nodes needs many more iterations than a code with higher minimum variable node degree. This is the case for codes with performance close to the capacity, which are known to need degree-2 variable nodes [6]. Moreover, due to the "threshold behavior" with respect to the minimum variable node degree, one can expect that good codes for low number of iterations would be similar to regular codes.

In the following, we perform LDPC code optimization for a low number of iterations and a low target BER. The obtained results shed light on the optimal code structure.

IV. Code Optimization

We jointly optimize the degree distributions of variable and check nodes using a semi-random walk algorithm [12]. Given a *target* BER and a number of iterations, the optimization algorithm can be described as follows:

1) the initial degree distributions (properly chosen) are loaded;
2) the channel is worsened (by increasing the erasure probability) until the BER upper bound in (1), computed using EXIT charts, is higher than the target BER;
3) new degree distributions (with fixed code rate) are generated considering small (random) perturbation of the previous degree distributions, until the corresponding BER is smaller than the previous;
4) if the maximum number of trials (set a priori) is reached the optimization process stops and the optimized degree distributions are obtained; otherwise, it restarts from step 2.

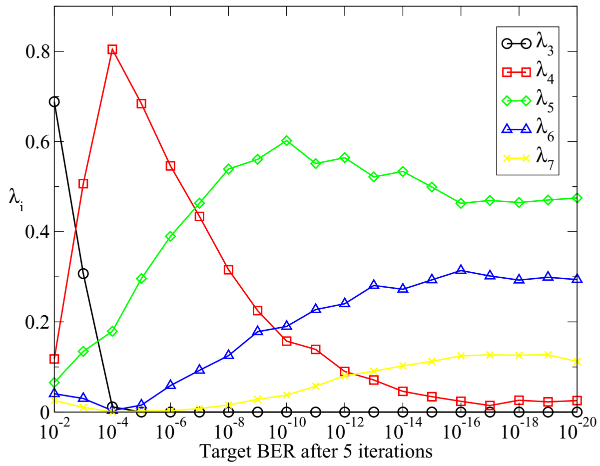

Fig. 2. Variable node degree distribution coefficients versus target BER considering 5 decoding iterations.

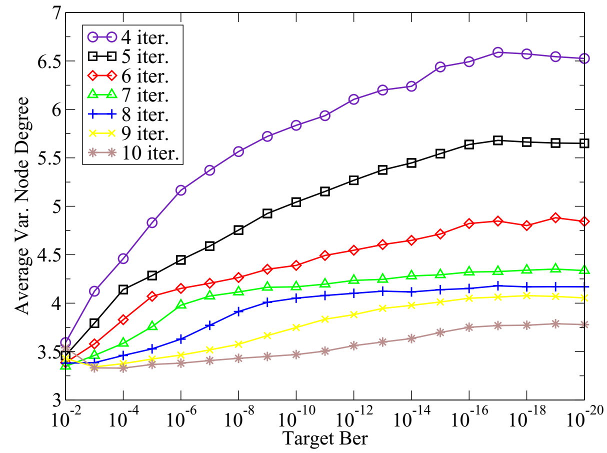

Fig. 4. LDPC graph connectivity (average variable node degree) versus target BER. Various numbers of decoding iterations are considered.

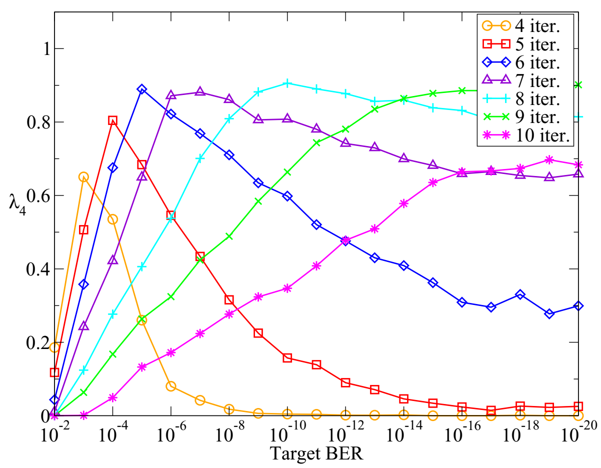

Fig. 3. Coefficient λ_4 of the variable node degree distribution shown as a function of the target BER. Various decoding iterations are considered.

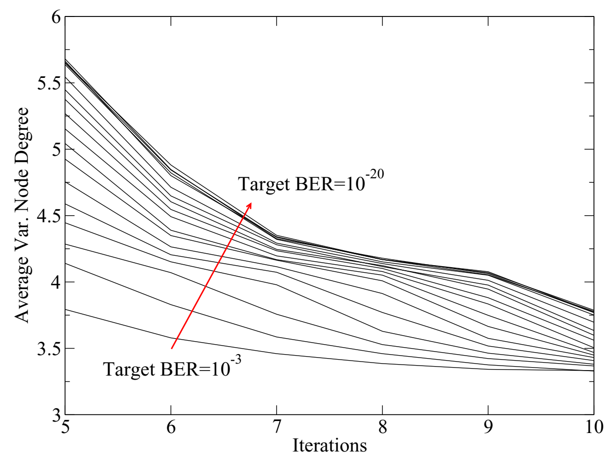

Fig. 5. LDPC graph connectivity (average variable node degree) versus decoding iteration number. Various values of the target BER $(10^{-3}, 10^{-4}, \ldots, 10^{-20})$ are considered.

V. Numerical Results

Considering a number of decoding iterations between 4 and 10 and several values of the target BER, 400 optimizations have been performed for each couple of iteration and target BER values. The obtained variable node degree distributions are obtained by averaging over the optimization results. The following degree distributions have been chosen as the initial set: $\lambda_3 = \ldots = \lambda_{12} = 0.1$ and $\rho_6 = \rho_8 = \rho_{10} = \rho_{12} = \rho_{14} = \rho_{16} = \rho_{18} = \rho_{20} = \rho_{22} = \rho_{24} = 0.1$. These distributions correspond to a rate-$1/2$ code ensemble.

In Fig. 2, the average optimized variable node degree distribution coefficients are shown as a function of the target BER, considering 5 decoding iterations. One can observe that as the target BER becomes lower the degree of the most significant coefficient increases.

In Fig. 3, the relationship between the coefficient λ_4, i.e., the fraction of edges in the LDPC code graph connected to degree-4 variable nodes, and the target BER is investigated. Several curves, corresponding to different number of decoding iterations are shown. Clearly, for any number of iterations, the coefficient λ_4 in the optimized degree distributions reaches a maximum close to 1 and then decays for decreasing values of the target BER. Moreover, one can notice that as the number of decoding iterations increases, the maximum moves to the right, suggesting that the evolution towards high degree variable nodes becomes slower.

In order to better understand the structure of an LDPC code designed for low number of iterations, we investigate the behavior of a function of the variable node degree distribution given by the ratio between the number of edges in the graph and the number of variable nodes, i.e., it corresponds to the *average variable node degree*. It can be shown that this quantity can be computed as $1/(\sum_i \lambda_i/i)$. In Fig. 4, the relationship of the average variable node degree with the target BER is shown, for various numbers of decoding iterations. The average variable node degree seems to be a monotonically increasing function of the target BER. Moreover, the lower the number of iterations, the higher is the average variable node degree. In Fig. 5, the relationship between the average variable node degree and the number of iterations is shown at different values of the target BER. The curves are monotonically decreasing. Interestingly, the

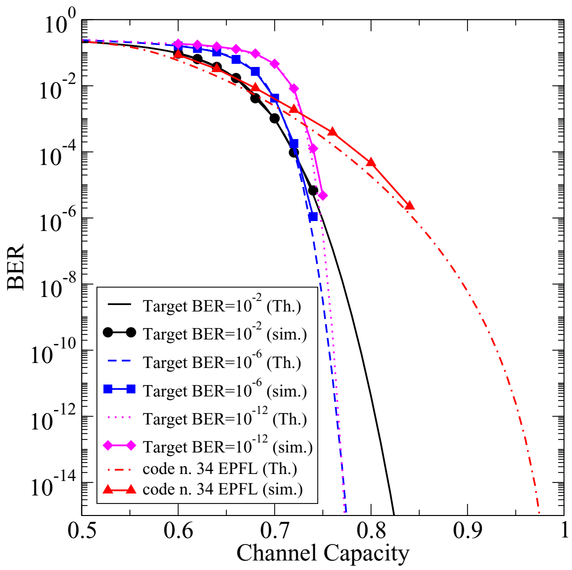

Fig. 6. Theoretical and simulation BER as a function of the BEC channel capacity for 4 codes. The number of decoding iterations is 5.

average variable node degree concentrates in a short range (between 3.3 and 3.7) as the number of iterations increases. More precisely, the trend of the curves suggests asymptotic convergence to a value close to 3.3, *regardless* of the target BER.

Considering 5 decoding iterations and target BERs equal to 10^{-2}, 10^{-6}, 10^{-12}, we chose the best obtained degree distributions in terms of channel capacity convergence threshold. LDPC codes with codeword length 6000 were then generated considering the selected degree distributions. As a reference, the rate-$1/2$ degree distributions number 34 in [13] (referred to as "EPFL"), with performance close to the AWGN channel capacity, have been considered, and an actual LDPC code has correspondingly been generated with codeword length 10000. In Fig. 6, the BER curves of the generated codes are shown, as functions of the BEC channel capacity, i.e., $1 - p_e$, in which p_e is the probability of erasure. In particular, for each code/degree distribution the figure shows (i) a theoretical BER curve obtained using EXIT charts and the upper bound in (1) and (ii) a BER curve obtained by Monte Carlo simulation of the actual code. In all cases, the number of iterations is 5. One can observe that, at BER$= 10^{-2}$ the "ad hoc optimized" code performs better than the codes optimized for target BER equal to 10^{-6} and 10^{-12}. The "EPFL" code exhibits better performance than the code optimized for target BER equal to 10^{-2}; this is due to the fact that in our optimization degree-2 variable nodes were omitted. At target BER equal to 10^{-6}, the best code is the ad hoc optimized code. One can observe that the theoretical BER curve is always lower than or equal to the simulated one, even if the theoretical curve is obtained using the "upper bound." This apparent contradiction is due to the fact that: (i) the actual codes contain cycles and (ii) for the particular case of BEC the upper bound is achieved.

Interestingly, the variable node degree distributions of the best optimized codes are characterized by the presence of a $\lambda_{i_{\min}}$ close to 1, i.e., these codes are similar to regular LDPC codes. Moreover, it is clear that codes optimized for a low target BER after a small number of iterations significantly differ from the "high performance" LDPC codes designed to reach the channel capacity. We remark that the presence of short cycles in the code graph is the major concern when very low target BER is desired. In particular, a low target BER calls for higher degree variable nodes, but the design of an LDPC code without short cycles with high variable node degrees can be troublesome.

VI. Concluding Remarks

In this paper, we have used an EXIT chart-based analysis in order to estimate the system BER and design LDPC codes optimized for a BEC considering a small number of decoding iterations. In particular, insights are given into the optimized LDPC code structure, showing that LDPC codes designed for low target BER after a small number of iterations (between 3 and 8) are similar to *regular* LDPC codes. The advantage of our approach, with respect to other approaches, such as density evolution-based optimization [6], is that application to other channels, with or without memory, is straightforward. We are currently working in this direction.

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