ARRAY SELF CALIBRATION: IDENTIFIABILITY ISSUES

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ABSTRACT

Array self calibration consists of identifying array shape distortions and deviations to gain and phase sensor responses, in an unknown source field. Conditions of local identifiability of these parameters are established, and turn out to depend on the type of array and the type of field. The minimal number of sources and sensors is calculated in each case, and the nature of the remaining degrees of freedom is interpreted. With an additional knowledge, that can be provided by a manœuvre or by a perfect sensor, it is shown that the latter parameters can be in turn identified.

1 Introduction

The subject of array calibration (and mainly the compensation for the effect of shape distortion) has been addressed by several authors in the literature. However, identifiability conditions have apparently been addressed only by Rockah and Schultheiss in [1] [2]. Their approach is valid except for some pathological cases, including unfortunately the linear array.

Many other works not addressing identifiabilty (but devising algorithms) include the works of Weiss and Friedlander [4], and Vezzosi [3]. In these approaches, conditions under which a good behaviour of the algorithm is to be expected are not accurately known. In [3], a theorem states that source vectors identifiability is guaranteed if there is at least twice more sensors than sources. This theorem will be referred to as Vezzosi's theorem.

In this paper, the results reported in [1] [2] are extended, and in particular apply to the linear array in both far- and near-fields. Let Γ be the covariance of observations in a given narrow band, and Υ the ideal form that should have the covariance if the array were subject to no distortion of any kind. Then one assumes throughout the paper that there exists a matrix G such that for any snapshot n:

$$\Gamma(n) = G\,\Upsilon(n)\,G^{\dagger}. \tag{1}$$

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Matrix G is unknown and accounts for gain and phase sensor deviations, that are wished to be estimated. If there is no coupling between sensors, matrix G is diagonal, which simplifies the problem; this will be subsequently assumed. In the present paper it will be given the conditions linking the number of snapshots, sensors, and sources, under which matrix G can be estimated.

2 Model, properties, and assumptions

Consider an array with K sensors, O its (arbitrary) origin, and $\boldsymbol{b}_k$ the vector defining the location of the $k-$th sensor from the origin. Assume there are N snapshots each having possibly different source scenarios with J sources. Now denote $R_{j,n}$ and $\theta_{j,n}$ the range and bearing of the $j-$th source impinging on the array, viewed from O in direction $\boldsymbol{u}_j$, in snapshot n.

The classical narrow band observation model writes

$$r(n) = \sum_{j=1}^{J} s_{j,n}\boldsymbol{d}_{j,n} + \boldsymbol{v}(n),\ 1 \leq n \leq N, \tag{2}$$

where n and j label snapshots and sources, respectively, $\boldsymbol{d}_{j,n} = \boldsymbol{d}(\theta_{j,n}; R_{j,n})$ are vectors with unit modulus entries, and $E\{\boldsymbol{v}(n)\,\boldsymbol{v}(n)^{\dagger}\} = \sigma\, I$.

If model (2) were exactly followed, the covariance of observations would take the ideal form:

$$\Upsilon(n) = \sum_{j=1}^{J} \gamma_{j,n}\, \boldsymbol{d}_{j,n}\, \boldsymbol{d}_{j,n}^{\dagger} + \sigma I. \tag{3}$$

The actual covariance matrix Γ is related to the ideal one through (1) if calibration errors are present; additionally, if the array shape is distorted, vectors $\boldsymbol{d}_{j,n}$ are affected, but (3) still holds true.

2.1 Limits to the nearness of the field

In a first approximation, variables j and n separate in the expression of directional vector $\boldsymbol{d}_{j,n}$, so that one can write the phase of its $k-$th component as:

$$\psi_k(j,n) = \boldsymbol{f}(k)^T \boldsymbol{g}(j,n), \tag{4}$$

where $\boldsymbol{f}(\cdot)$ and $\boldsymbol{g}(\cdot)$ are known vector functions whose dimension depends on the array considered (see subsequent sections). In fact, at pulsation ω, the phase difference between the $j-$th source contribution received at the origin and that received on the $k-$th sensor can be approximated, up to second order in $||\boldsymbol{b}_k||/R_{j,n}$, by:

$$\Psi_k(j,n) \approx \frac{\omega}{C}\left(\boldsymbol{b}_k \cdot \boldsymbol{u}_{j,n} - \frac{1}{2R_{j,n}}||\boldsymbol{b}_k \wedge \boldsymbol{u}_{j,n}||^2\right). \quad (5)$$

2.2 Sensor gains

Sensor gains (*i.e.* moduli of the entries of G) are easily identified. In fact, since G is diagonal, one obviously obtains from (1) and (3) that $|G_{kk}|^2(\sum_j \gamma_{j,n} + \sigma) = \Gamma_{kk}(n)$, which determines sensor gains, up to an arbitrary scale factor.

2.3 Bound on the number of sources

Now denote by Φ the phase components of G, and build the normalized covariance matrix

$$\bar{\Gamma}(n) = D(n)\,\Gamma(n)\,D(n)^\dagger,\ D(n) = Diag^{1/2}\{\Gamma(n)\}.$$

so that $\bar{\Gamma} = (\sum_j \gamma_j + \sigma)^{-1}\Phi\,\Upsilon\,\Phi^\dagger$ for any snapshot n (here omitted for conciseness).

Since the noise component of $\bar{\Gamma}$ is still proportional to identity, Vezzosi's theorem tells that directional vectors $\boldsymbol{d}'_{j,n} = \Phi \boldsymbol{d}_{j,n}$ can be identified uniquely up to a scalar phase factor provided that $K > 2\,J$ for every n. The choice of N needs to be made accordingly.

Without loss of generality, one can impose $\Phi_{11} = 1$ and $\psi_1(j,n) = 0$. Because of (4) and with obvious notation, the phase of $\boldsymbol{d}'_{j,n} = \Phi\,\boldsymbol{d}_{j,n}$ takes the form:

$$\psi'_k(j,n) = \boldsymbol{f}(k)^T\boldsymbol{g}(j,n) + \phi_k, \quad (6)$$

which shows that only the total number of sources observed over the N snapshots counts: *using one snapshot with NJ sources is equivalent to using N snapshots each with J sources.* Note that in both cases, sources need to have different locations. Thus from now, it will be considered that there is a single snapshot and JN sources; index n is dropped, and j varies between 1 and JN.

Now, the identification of sensor phases and locations is entirely summarized by system (6).

2.4 Local identifiability of sensor phases

Assuming the gap between actual and ideal arrays is small, it is possible to expand the phase (6). In fact, vary index k from 2 to K, and store the values of ψ' into a $K-1 \times JN$ matrix:

$$\Psi' = X\,Y^T,\ X_{k:} \stackrel{\text{def}}{=} [\boldsymbol{f}(k)^T, \phi_k],\ Y_{j:} = [\boldsymbol{g}(j),\,1] \quad (7)$$

The solution of (7) is rather difficult because of its nonlinearities. However, if array and source parameters are approximately known, we have intitial guesses X_0 and Y_0 for X and Y. A first order expansion yields:

$$\mathcal{C} \stackrel{\text{def}}{=} \Psi' - X_0\,Y_0^T \approx \mathcal{X}\,Y_0^T + X_0\,\mathcal{Y}^T, \quad (8)$$

defining $\mathcal{X} = X - X_0$ and $\mathcal{Y} = Y - Y_0$ (calligraphic symbols represent small quantities). Local identifiability is insured if small deviations $\mathcal{X}$ and $\mathcal{Y}$ can be indeed identified. Generically, Y_0 and X_0 are full rank, and the lemma stated in the next section can be applied (but there are exceptions with null probability).

2.5 Algebraic Lemma

Let A, B, and $\mathcal{C}$ be three matrices of dimension $JN \times q$, $K-1 \times p$ and $K-1 \times JN$, respectively, and consider the following equation in the unknowns $\mathcal{X}$ and $\mathcal{Y}$, of dimension $K-1 \times q$ and $JN \times p$:

$$\mathcal{X}A^T + B\mathcal{Y}^T = \mathcal{C}. \quad (9)$$

The purpose of this section is to give the solution of this linear system. By an obvious postmultiplication, we get from (9) the relations:

$$\begin{aligned}\mathcal{X}\,A^TA &= \mathcal{C}\,A - B\,\mathcal{Y}^TA,\\ \mathcal{Y}\,B^TB &= \mathcal{C}^TB - A\,\mathcal{X}^TB.\end{aligned}$$

For any full column rank matrix F, denote $\Pi_F \stackrel{\text{def}}{=} I - F(F^TF)^{-1}F^T$. If matrices A and B are full-rank, this system can be solved by backsubstitution yielding:

$$\begin{aligned}\Pi_B\,[\mathcal{X} - \mathcal{C}\,A(A^TA)^{-1}] = 0,\\ \Pi_A\,[\mathcal{Y} - \mathcal{C}^TB(B^TB)^{-1}] = 0.\end{aligned}$$

It is convenient to rewrite this solution in the following manner

$$\mathcal{X} = \mathcal{C}\,A(A^TA)^{-1} + B\,M_x, \quad (10)$$

$$\mathcal{Y} = \mathcal{C}^TB(B^TB)^{-1} + A\,M_y, \quad (11)$$

where M_x and M_y are unknown matrices of respective dimension $p \times q$ and $q \times p$. Now let us look whether M_x and M_y are subject to any additional constraints. If expressions (10) and (11) are plugged back into (9), it can be deduced after pre- and post-multiplication by $(B^TB)^{-1}B^T$ and $A(A^TA)^{-1}$ that:

$$\mathcal{S} \stackrel{\text{def}}{=} (B^TB)^{-1}B^T\,\mathcal{C}\,A(A^TA)^{-1} = -M_x - M_y^T. \quad (12)$$

This indeed shows that M_x and M_y are not independent to each other. Both matrices can be parametrized by a single $p \times q$ matrix, Δ, as:

$$M_x = -\frac{1}{2}\mathcal{S} + \Delta, \quad M_y = -\frac{1}{2}\mathcal{S}^T - \Delta^T. \quad (13)$$

It is to be retained from this lemma that, because of (10), (11), and (13), and with obvious notation:

$$\mathcal{X} = \mathcal{X}_s + B\Delta, \quad \mathcal{Y} = \mathcal{Y}_s - A\Delta^T. \quad (14)$$

Matrix Δ induces indeterminacies, and the array can be calibrated only if it can be determined by other means, as commented in the next sections.

Another constraint concerns matrix $\mathcal{C}$. In fact, plugging back (10), (11) and (12) into (9) yields:

$$\Pi_B\,\mathcal{C}\,\Pi_A = 0. \quad (15)$$

This last equation expresses the compatibility conditions that allow (9) to admit at least one solution.

One can show that if (7) is solved in the LS sense, then (15) is not necessarily satisfied, but the solution is still given by (14).

2.6 Dependency between variables

It happens that because of the field or array type, entries of $\mathcal{X}$ or $\mathcal{Y}$ are not free. Therefore, it is convenient to introduce new variables with free entries, $\mathcal{W}$ and $\mathcal{Z}$:

$$\mathcal{X}_{k:} = \mathcal{W}_{k:} E_k, \quad \mathcal{Y}_{j:} = \mathcal{Z}_{j:} F_j, \tag{16}$$

where E_k and F_j are of dimension $q' \times q$ and $p' \times p$, respectively, with $q' \leq q$ and $p' \leq p$.

3 Application to real-world arrays

In far fields, we have that

$$X_{k:} = [\frac{\omega}{C} x_k, \frac{\omega}{C} y_k, \phi_k], \text{and} Y_{j:} = [\cos\theta_j, \sin\theta_j, 1].$$

In near fields, equation (5) leads to the definitions

$$X_{k:} = [\tfrac{\omega}{C} x_k, \tfrac{\omega}{C} y_k, \tfrac{\omega}{C}\tfrac{1}{2} x_k^2, \tfrac{\omega}{C}\tfrac{1}{2} y_k^2, \tfrac{\omega}{C} x_k y_k, \phi_k],$$
$$Y_{j:} = [\cos\theta_j, \sin\theta_j, -\tfrac{\sin^2\theta_j}{R_j}, -\tfrac{\cos^2\theta_j}{R_j}, \tfrac{\sin\theta_j\cos\theta_j}{R_j}, 1].$$

3.1 Far-field Linear Array Problem (LAP)

After a straightforward manipulation, on finds that $A_{j:} = [\cos\theta_{j0}\ \sin\theta_{j0}\ 1]$, and $B_{k:} = \frac{\omega}{C} x_{k0}$ since the initial guess is $y_{k0} = 0$. In other words, $p = 1$ and $q = 3$.

The unknown perturbations to be estimated are $\mathcal{X}_{k:} = [\frac{\omega}{C} dx_k\ \frac{\omega}{C} dy_k\ d\phi_k]$ and $\mathcal{Y}_j = -d\theta_j \sin\theta_{j0}$. There are thus $pq = 3$ indeterminacies that can be explicited thanks to (14). Since $d\boldsymbol{y}_0 = 0$, the first one is a dilation along the Ox axis: $d\boldsymbol{x} - d\boldsymbol{x}_s = \Delta_1 \boldsymbol{x}_0$; the second one is a rotation of the array about its origin: $d\boldsymbol{y} - d\boldsymbol{y}_s = \Delta_2 \boldsymbol{x}_0$; the third one is a spatial linear phase: $d\boldsymbol{\phi} - d\boldsymbol{\phi}_s = \frac{\omega}{C}\Delta_3 \boldsymbol{x}_0$. The relation $\mathcal{Y} - \mathcal{Y}_s = -A\Delta^T$ expresses the same indeterminacies in the (dual) source space: $d\theta_j \sin\theta_{j0} = \Delta_1 \cos\theta_{j0} + \Delta_2 \sin\theta_{j0} + \Delta_3$.

A sufficient condition for $A^T A$ to be invertible is that at least 3 sources with different bearings (modulo π) are present. Next, $B^T B$ is invertible as soon as $K > 1$.

3.2 Far-field Plane Array Problem (PAP)

Now it is considered that sensors and sources belong to the same plane Oxy. Compared to the previous section, A remains the same and $q = 3$, but y_{k0} is not null anymore, so that $B_{k:} = [\frac{\omega}{C} x_{k0}, \frac{\omega}{C} y_{k0}]$, $\mathcal{Y}_{j:} = d\theta_j F_j$, $F_j = [-\sin\theta_{j0}, \cos\theta_{j0}]$. According to our remarks of section 2.6, $p = 2$ but there is only $p' = 1$ degree of freedom.

Thanks to the result of our lemma, (14), a first order perturbation analysis yields that $\mathcal{X} - \mathcal{X}_s = B\Delta$ and $\mathcal{Y} - \mathcal{Y}_s = -A\Delta^T$, where Δ is a 3×2 unknown matrix. This is true provided both A and B are full rank. The former is full rank if at least 3 sources are present (as in the linear array case), and B looses its rank if the array degenerates into a linear one (*i.e.* $K > 2$ is necessary)..

The actual number of unknowns in Δ is much smaller than $pq = 6$. In fact, the relations between entries of Δ can be obtained by noticing that $\mathcal{Y}_{j1}\cos\theta_{j0} + \mathcal{Y}_{j2}\sin\theta_{j0} = 0$ for every j; now using (14), we obtain that $(\mathcal{Y}_s - A\Delta^T)[\cos\theta_{j0}\ \sin\theta_{j0}]^T = 0, \forall j$.

After some manipulation, this yields:

$$[\cos^2\theta_{j0}, \sin\theta_{j0}\cos\theta_{j0}, \sin^2\theta_{j0}, c\cos\theta_{j0}, \sin\theta_{j0}]$$
$$\cdot [\Delta_{11}, \Delta_{12} + \Delta_{21}, \Delta_{22}, \Delta_{13}, \Delta_{23}]^T = \mathcal{Y}_s \begin{bmatrix} \cos\theta_{j0} \\ \sin\theta_{j0} \end{bmatrix}.$$

Thus, if there are at least 5 (distinct) sources present, the above system is full rank, and the six entries in Δ can be determined up to an additive plane rotation:

$$\Delta = \Delta_s + \begin{bmatrix} 0 & \varepsilon & 0 \\ -\varepsilon & 0 & 0 \end{bmatrix}.$$

3.3 Far-field Volume Array Problem (VAP)

Here the general case is considered: sensors and sources can be located anywhere in the space. Assuming a sperical coordinate system for the sources, we end up with:

$$B_k = \frac{\omega}{C}[x_{k0}, y_{k0}, z_{k0}],$$
$$F_j = -\begin{bmatrix} \sin\theta_{j0}\cos\varphi_{j0} & -\cos\theta_{j0}\cos\varphi_{j0} & 0 \\ \cos\theta_{j0}\sin\varphi_{j0} & \sin\theta_{j0}\sin\varphi_{j0} & -\cos\varphi_{j0} \end{bmatrix},$$
$$A_{j:} = [\cos\theta_{j0}\cos\varphi_{j0}, \sin\theta_{j0}\cos\varphi_{j0}, \sin\varphi_{j0}, 1].$$

Now the kernel of matrix F_j is spanned by $F_j^\perp = [\cos\theta_j \cos\varphi_j, \sin\theta_j\cos\varphi_j, \sin\varphi_j]^T$. Then with similar arguments as in the previous section, the quantity $A_{j:}\Delta^T F_j^\perp$ is equal to the known quantity $\mathcal{Y}_{:js} F_j^\perp$. This leads after some manipulations to the linear system:

$$\begin{bmatrix} \cos^2\theta\cos^2\varphi \\ \cos\theta\sin\theta\cos^2\varphi \\ \cos\theta\cos\varphi\sin\varphi \\ \sin^2\theta\cos^2\varphi \\ \sin\theta\cos\varphi\sin\varphi \\ \sin^2\varphi \\ \cos\theta\cos\varphi \\ \sin\theta\cos\varphi \\ \sin\varphi \end{bmatrix}^T \cdot \begin{bmatrix} \Delta_{11} \\ \Delta_{12}+\Delta_{21} \\ \Delta_{13}+\Delta_{31} \\ \Delta_{22} \\ \Delta_{23}+\Delta_{32} \\ \Delta_{33} \\ \Delta_{14} \\ \Delta_{24} \\ \Delta_{34} \end{bmatrix} = \mathcal{Y}_{:js} F_j^\perp.$$

As a consequence, matrix Δ is known up to 3 degrees of freedom in the form of an infenitesimal rotation:

$$\Delta = \Delta_s + \begin{bmatrix} 0 & \varepsilon & \eta & 0 \\ -\varepsilon & 0 & \delta & 0 \\ -\eta & -\delta & 0 & 0 \end{bmatrix}.$$

Finally, note that matrix B is rank deficient as soon as the array is plane ($K > 3$ necessary), and at least 4 sources are needed for A to be full rank.

3.4 Near-field LAP

In near field, we must increase q up to $q = 5$ in order to preserve the separation of variables, even if the number of free parameters is still $q' = 3$. In fact:

$$A_{j:} = \left[\cos\theta_{j0}, \sin\theta_{j0}, \frac{-\sin^2\theta_{j0}}{R_{j0}}, \frac{\sin\theta_{j0}\cos\theta_{j0}}{R_{j0}}, 1\right],$$
$$\mathcal{X}_{k:} = \mathcal{W}_{k:} E_k,$$

$$\mathcal{W}_{k:} = \left[\frac{\omega}{C}dx_k,\ \frac{\omega}{C}dy_k,\ d\phi_k\right]$$

$$E_k = \begin{bmatrix} 1 & 0 & x_{k0} & 0 & 0 \\ 0 & 1 & 0 & x_{k0} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next, the number of free source parameters has increased to $p = 2$ and:

$$\begin{aligned} B_{k:} &= \frac{\omega}{C}\left[x_{k0}\ x_{k0}^2\right], \\ \mathcal{Y}_{j:} &= \mathcal{Z}_{j:} F_j, \\ \mathcal{Z}_{j:} &= \left[d\theta_j\ dR_j\right], \\ F_j &= \begin{bmatrix} -\sin\theta_{j0} & \frac{-\sin\theta_{j0}\cos\theta_{j0}}{R_{j0}} \\ 0 & \frac{\sin^2\theta_{j0}}{2R_{j0}^2} \end{bmatrix}. \end{aligned}$$

Now to interpret the nature of indeterminacies, consider the matrix spanning the kernel of E_k:

$$E_k^{\perp} = \begin{bmatrix} -x_{k0} & 0 \\ 0 & -x_{k0} \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now since $\mathcal{X}_{k:}E_k^{\perp} = 0$, we have that $\mathcal{X}'_{sk:} \stackrel{\text{def}}{=} \mathcal{X}_{sk:}E_k^{\perp} = -B_{k:}\Delta E_k^{\perp}$. Denoting Δ_{ab} the entries of the 2×5 matrix Δ, this can be rearranged into:

$$\mathcal{X}'_{sk:} = \left[x_{k0}\ x_{k0}^2\ x_{k0}^3\right] \begin{bmatrix} \Delta_{13} & \Delta_{14} \\ -\Delta_{11} + \Delta_{23} & -\Delta_{12} + \Delta_{24} \\ -\Delta_{21} & \Delta_{22} \end{bmatrix},$$

where the left-hand side is known. Thus by taking sufficiently many values of k, one can hope to be able to solve this overdetermined system for Δ_{13}, Δ_{14}, Δ_{21}, and Δ_{22}. This is the case if K–$1 \geq 3$ (with different abscisses x_{k0}), which implies not only that $B^T B$ is regular, but also that the above Van der Monde system is at least of rank 3.

As a conclusion, there remain four degrees of freedom that cannot be fixed, among the $pq = 10$ entries of Δ, and it is possible to interpret them by plugging back the results in (14). The two first indeterminacies are of the same nature as in the far-field case:

$$\begin{aligned} dx_k - dx_{sk} - \Delta_{21}x_{0k}^2 &= \Delta_{11}x_{0k}, \\ dy_k - dy_{sk} - \Delta_{22}x_{0k}^2 &= \Delta_{12}x_{0k}, \\ d\phi_k - d\phi_{sk} &= \frac{\omega}{C}(\Delta_{15}x_{ok} + \Delta_{25}x_{25}^2). \end{aligned}$$

Indeed, the first is a dilation and the second a rotation. On the other hand, the third one is different since there is now an infinitesimal quadratic term in the phase. The relation $\mathcal{Y} - \mathcal{Y}_s = -A\Delta^T$ would yield now $p = 2$ equations.

Lastly, for the lemma to be applicable, A and B need to be full rank. Therefore, it is necessary to have at least $K \geq 4$ sensors, and it is sufficient to have at least $JN \geq 5$. Comparing to the far-field linear array, we need 2 additional sources and 2 additional sensors.

3.5 Near-field SAP

In near-field, one can show that $A_{j:} = [cos\theta_j,\ \sin\theta_j,\ -\sin^2\theta_j/R_j,\ \sin\theta_j\cos\theta_j/R_j,\ -\cos^2\theta_j/R_j,\ 1]$, and $B_{k:} = [x_k,\ y_k,\ x_k^2,\ x_k y_k,\ y_k^2]$. Thus, B is full rank if the array does not lie on a conic. A complete discussion of this case is postponed to a subsequent paper.

4 Concluding remarks

As a conclusion, it is important to explain how we can use these results in order to cope with indeterminacies. In all array cases, there remains generally a residual plane rotation that can be determined only by resorting to the exact knowledge of the bearing of an additional sensor.

In the far-field LAP problem, either 1 manœuvre, or 1 additional sensor with known $(dx,\ dy,\ d\phi)$, determines the dilation and linear phase indeterminacies.

In the near-field LAP, either 1 manœuvre, or 2 additional sensors with known $(dx,\ dy,\ d\phi)$ for the first but only $d\phi$ for the second, determine completely the dilation and the 2 parameters of the quadratic phase indeterminacies.

In the far-field PAP, only the residual rotation in the plane remains. On the other hand, if a manœuvre is performed, then one can determine its angle even if the residual rotation is unknown.

In the far-field VAP, 1 manœuvre in the canonical plane (constant elevation) fixes two degrees of freedom, but leaves undetermined a residual rotation in that plane. This manœuvre can be replaced by the knowledge of the elevation angles $d\varphi$ of 2 additional sensors.

In all cases, one can also exploit a source trajectory model in place of manœuvres.

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