A GENERALIZED CORRELATION FUNCTION FOR MAGNIFIED/REDUCED SIGNALS

Axel Busboom, Hans Dieter Schotten, and Harald Elders–Boll
Institut für Elektrische Nachrichtentechnik
RWTH Aachen, D–52056 Aachen, Germany
Tel: +49 241 807678; fax: +49 241 8888196
e-mail: busboom@ient.rwth-aachen.de

ABSTRACT

A generalization of the correlation function is explored which, besides a relative time shift between the signals to be correlated, also takes into account different scalings on the time axis (i.e., magnification/reduction). It is shown how the generalized correlation function for continous signals can be sampled and computed without loss of information and thus can be described by discrete–time signals. Envisaged applications comprise coded aperture imaging, measurement, radar, and digital communications. Special attention is paid to tomographic imaging using coded apertures. It is demonstrated how individual slices of an object can be reconstructed by correlating the recorded image with suitably designed decoding filters using the generalized correlation function.

1 INTRODUCTION

In many applications the detection of signals with a known shape in the presence of noise is necessary where the signal to be detected may not only exhibit an unknown delay but also a dilation (or diminution) on the time axis. For a mere time shift, the crosscorrelation between the noisy signal and the original signal shape can be used for signal detection and measurement of the delay. In order to account for an additional scaling on the time axis, we define and examine a generalized correlation function which depends on the time shift τ and a magnification factor λ. In section 2, the generalized correlation is defined and its properties are discussed. In particular, it is shown how the generalized correlation can be sampled and computed on a finite grid without loss of information.

In section 3, the application of the generalized correlation function to image reconstruction in coded aperture imaging is presented. In coded aperture imaging, the image of a point source is a shifted and magnified shadow of the aperture where the magnification factor corresponds to the distance of the source from the aperture mask. Using decoding filters pertaining to the different magnification factors, i.e., to the different object distances, individual slices of the imaged object can be reconstructed via the generalized correlation function.

2 DEFINITION AND PROPERTIES OF THE GENERALIZED CORRELATION

2.1 Definition

We define the generalized crosscorrelation function (GCCF) between two continuous–time signals $s(t)$ and $g(t)$ as

$$\phi_{sg}(\tau,\ \lambda) = \int_{-\infty}^{\infty} s^*(t)\ \frac{1}{\sqrt{\lambda}} g\left(\frac{t+\tau}{\lambda}\right)\ dt, \quad \lambda > 0 \quad (1)$$

where τ denotes the shift as in the conventional correlation function while λ is the time axis magnification. The factor $1/\sqrt{\lambda}$ normalizes the energy of the magnified/reduced version of $g(t)$. Note that for $\lambda = 1$ equation (1) yields the conventional crosscorrelation function [1]. If s and g are identical, the expression in equation (1) will be referred to as the generalized autocorrelation function (GACF).

2.2 Properties of the GACF/GCCF

In this section, some useful properties of the GACF/GCCF are presented. Let $S(f)$ and $G(f)$ denote the Fourier transforms of $s(t)$ and $g(t)$, respectively. Then the Fourier transform of $\phi_{sg}(\tau,\ \lambda)$ for a fixed scaling factor λ is given by

$$\Phi_{sg}(f,\ \lambda) = \sqrt{\lambda} S^*(f) G(f\lambda). \quad (2)$$

The GCCF exhibits the symmetry property

$$\phi_{sg}(\tau,\ \lambda) = \phi^*_{gs}\left(-\frac{\tau}{\lambda}, \frac{1}{\lambda}\right). \quad (3)$$

Furthermore, $\phi_{sg}(\tau,\ \lambda)$ can easily be shown to be bounded by

$$|\phi_{sg}(\tau,\ \lambda)| \leq \sqrt{E_s E_g}. \quad (4)$$

The "area" of the GCCF for a fixed λ is given by

$$\int_{-\infty}^{\infty} \phi_{sg}(\tau,\ \lambda)\ d\tau = \sqrt{\lambda} \int_{-\infty}^{\infty} s^*(t)\ dt \int_{-\infty}^{\infty} g(t)\ dt. \quad (5)$$

Hereafter, we will assume the signals $s(t)$ and $g(t)$ to be real valued.

2.3 Computation of the GACF/GCCF for Discrete–Time Signals

The case that s and g are discrete–time signals is of special interest for most applications. In order to obtain a continuous–time model for these signals, we assume that $s(t)$ and $g(t)$ are derived from discrete–time sequences $s_d(n)$ and $g_d(n)$ by passing them through sample–and–hold units, i.e.,

$$s(t) = \sum_{n=0}^{N-1} s(n)\,\mathrm{rect}\left(t-n-\frac{1}{2}\right), \tag{6}$$

$$g(t) = \sum_{n=0}^{M-1} g(n)\,\mathrm{rect}\left(t-n-\frac{1}{2}\right) \tag{7}$$

where

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if} \quad |t| \le 1/2 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

In order to compute the GACF between $s(t)$ and $g(t)$, it is convenient to define the trapezoidal–shaped basis function

$$\mathrm{Tz}(\tau,\ \lambda) = \int_{-\infty}^{\infty} \mathrm{rect}(t)\,\mathrm{rect}\left(\frac{t+\tau}{\lambda}\right)\,dt \tag{9}$$

whose shape is depicted in figure 1. With (1), (6), (7), and (9), we get

$$\phi_{sg}(\tau,\ \lambda) = \frac{1}{\sqrt{\lambda}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} s(n) g(m)$$

$$\times\,\mathrm{Tz}\left(\tau + \left(n+\frac{1}{2}\right) - \lambda\left(m+\frac{1}{2}\right),\ \lambda\right) \tag{10}$$

where the limits of summation can be further constrained by noting that $\mathrm{Tz}(\tau,\ \lambda) = 0$ for $|\tau| \ge (\lambda+1)/2$. With (10) the GACF/GCCF can be numerically computed in a finite number of steps for any τ and λ. It can easily be shown that the expression in (10) is continuous for $\lambda > 0$, vanishes for $\tau \ge \lambda M$ or $\tau \le -N$, and tends to zero for $\lambda \to 0$ or $\lambda \to \infty$.

2.3.1 *Computation for a Fixed λ*

For a fixed scaling factor λ, the expression in (10) is the finite sum of continuous and piecewise linear basis functions. Hence, $\phi_{sg}(\tau,\ \lambda)$ is also continuous and piecewise linear. For the computation of the GACF/GCCF, it is therefore sufficient to compute $\phi_{sg}(\tau,\ \lambda)$ for a finite number of shifts $\hat{\tau}$, namely

$$\hat{\tau}_{ij} = \lambda j - i, \quad i = 0(1)N, \quad j = 0(1)M. \tag{11}$$

All other values of the GACF/GCCF can be exactly determined by linear interpolation between the values $\phi_{sg}(\hat{\tau}_{ij},\ \lambda)$. Furthermore, with these values it is possible to exactly compute important parameters of the GACF/GCCF such as the maximum sidelobe value or the total sidelobe energy.

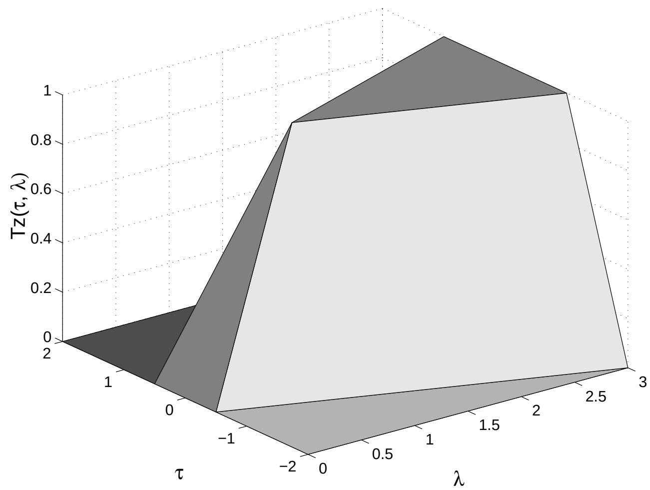

Figure 1: Trapezoid function.

2.3.2 *Computation for Arbitrary λ*

If a signal of unknown shift and magnification is to be detected, the GCCF must be evaluated over a range of values for both τ and λ. In this case, it is convenient to first consider the GCCF without the normalization factor, i.e., $\sqrt{\lambda}\phi_{sg}(\tau,\ \lambda)$. This expression, with (10), is the finite sum of Tz–functions which are continuous and piecewise linear in both τ and λ. Therefore, it is also continuous and piecewise linear, the edges between the linear regions given by (11). The vertices between these edges can be written as

$$\begin{aligned} \lambda &= \frac{i_1 - i_2}{j_1 - j_2}, & \tau &= \frac{i_1 j_2 - i_2 j_1}{j_1 - j_2}, \\ i_1 &= 1 \dots N, & i_2 &= 0 \dots i_1 - 1, \\ j_1 &= 1 \dots M, & j_2 &= 0 \dots j_1 - 1. \end{aligned} \tag{12}$$

This grid of edges and vertices is shown in figure 2 for $N = M = 5$.

If $\sqrt{\lambda}\phi_{sg}(\tau,\ \lambda)$ is known for the vertices given by (12), the GACF can be computed for any τ and λ by linearly interpolating $\sqrt{\lambda}\phi_{sg}(\tau,\ \lambda)$ and dividing by $\sqrt{\lambda}$.

Furthermore, it is usually important to know the locations and magnitudes of any local maxima of the magnitude $|\phi_{sg}(\tau,\ \lambda)|$ of the GACF/GCCF. As shown in the appendix, the only candidates for such local maxima are the grid vertices given by (12). Therefore, for most applications it will suffice to evaluate the GACF/GCCF at these vertices only.

3 APPLICATION TO CODED APERTURE IMAGING

As an application of the GACF/GCCF, we will now consider imaging systems with coded apertures. Coded aperture imaging is often used to increase the quantum collection efficiency for imaging high–energy photon sources, e.g., in X–ray and gamma–ray astronomy

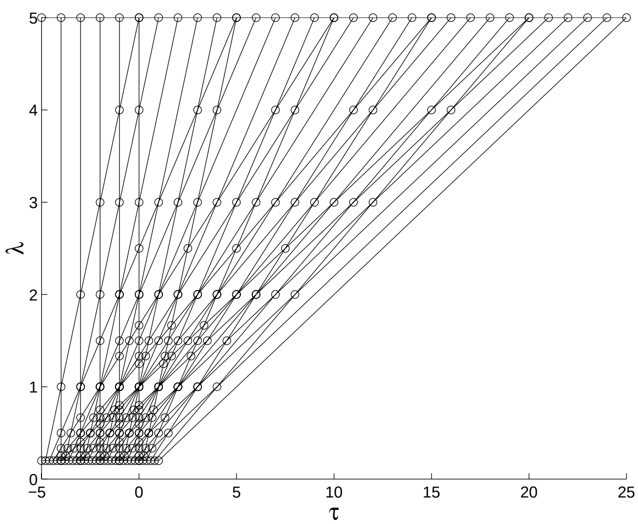

Figure 2: Grid showing the areas in which the unnormalized GCCF is linear.

or in nuclear medicine. However, it can also be utilized to obtain three–dimensional tomographic information of an imaged object from only a single or very few projections [2–6].

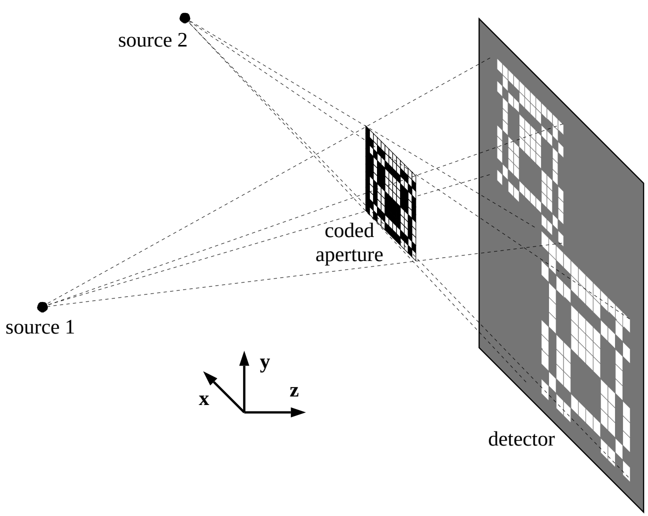

Figure 3: Imaging system with coded aperture.

A coded aperture imaging system is depicted in figure 3: a mask of transparent and opaque elements is placed in front of a planar detector. Apparently, a point source will cast a shadow on the detector which has the shape of the coded aperture, whose location corresponds to the x– and y–coordinates of the source, and whose magnification factor depends on the distance of the source from the aperture.

If a planar object parallel to the aperture plane rather than a single point source is imaged, then the detector image can be represented as the correlation of the two–dimensional object function with a magnified or reduced version of the aperture, the magnification factor being a function of the distance of the object plane from the aperture. Finally, if an extended, three–dimensional object is imaged, the detector image can be thought of as the superposition of the correlations of the individual object slices parallel to the aperture with differently magnified versions of the aperture. To a certain extent, it is then possible to reconstruct individual layers of the extended object from its coded image by filtering the detector image with suitably designed decoding filters for each object layer.

For a given slice, the decoding filter must undo the effect of the correlation of the object function with the aperture, i.e., it must approximate the inverse filter of the aperture mask, magnified with the respective scaling factor. The point spread function of the imaging system can then be expressed as the generalized crosscorrelation between the magnified aperture and the decoding filter where the object layers are uniquely characterized by their corresponding magnification factors λ.

There are several conceivable ways to design the decoding filters: Some approaches minimize the mean–squared error between the actual and the ideal, impulse–like point spread function [7, 8] while others aim at minimizing the maximum absolute error, leading to dynamic programming problems [9, 10]. The filter design can be extended by taking into account the attenuation of adjacent object layers by postulating that the GCCF between any out–of–focus layers and the decoding filter be small for all shifts. With sufficiently large filter sizes, acceptable attenuation can be obtained. Figure 5 shows an example of a GCCF between a 31 by 31 tap decoding filter and the aperture array depicted in figure 4 which is a nonredundant point distribution according to [2], for a scaling factor of $\lambda = 2/3$.

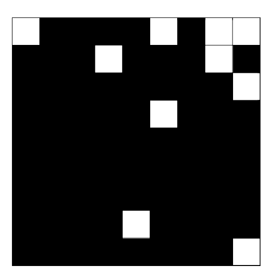

Figure 4: 9 by 9 aperture array (nonredundant point distribution according to [2]).

4 CONCLUSIONS

We have introduced and discussed a generalization to the correlation function which, as an additional parameter, depends on the time–axis magnification or reduc-

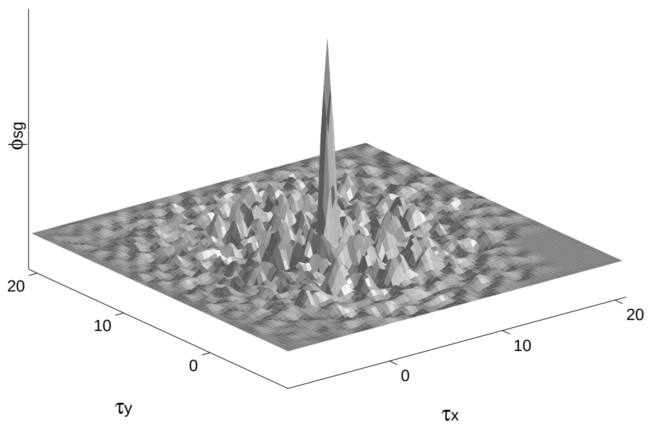

Figure 5: Generalized crosscorrelation between the aperture array from figure 4 and a 31 by 31 decoding filter.

tion of one of the signals to be correlated. It is an interesting result of our study that, for discrete–time signals, the GACF/GCCF can be completely determined by sampling it at a finite number of grid points in the τ–λ–plane. Also, any local maxima of the magnitude of the GACF/GCCF can only occur at these grid points.

However, the resulting sampling grid is not a rectangular one, and even for a fixed magnification factor λ the resulting sampling of τ is not equally spaced, except for well–behaved λ which can be expressed as the quotient of small integers (e.g. 2/3, 3/4, etc.).

While we believe that there may be applications in other areas such as measurement, radar, or digital communications, we are mainly concerned with the application of the two–dimensional GACF/GCCF to coded aperture imaging where the magnification factor λ is a function of the distance of the imaged object. We have successfully employed the generalized correlation function to reconstructing layers of an object from its coded image.

APPENDIX

In the following we prove that the magnitude $|\phi_{sg}|$ of the GACF/GCCF can take on its local maxima only at the grid vertices according to (12).

Inside any polygon of the grid (cf. figure 2), $\sqrt{\lambda}\phi_{sg}(\tau,\ \lambda)$ is a linear function of τ and λ; therefore we can write

$$\phi_{sg}(\tau,\ \lambda) = \frac{a\tau + b}{\sqrt{\lambda}} + c\sqrt{\lambda} \qquad (13)$$

where a, b, and c are real constants. For a local extremum, the partial derivative of ϕ_{sg} with respect to τ, $\partial\phi_{sg}/\partial\tau = a/\sqrt{\lambda}$, must vanish which implies $a = 0$, i.e. ϕ_{sg} depends only on λ. Therefore, local extrema can only occur on the boundary of the polygon.

Along any boundary between two polygons, we can write

$$\phi_{sg}(\tau,\ \lambda) = \frac{d}{\sqrt{\lambda}} + e\sqrt{\lambda} \qquad (14)$$

with real constants d and e. Taking the derivative with respect to λ and setting it to zero yields $\lambda = d/e$ for a local extremum. By analyzing the second derivative with respect to λ for this value and noting that $\lambda > 0$, we find that this local extremum can either be a minimum with a positive values of ϕ_{sg} or a maximum with a negative function value. Both cases yield local minima for $|\phi_{sg}|$ so that any local maxima can only occur at the grid vertices. □

ACKNOWLEDGMENT

This work was supported in part by grant FKP0015901B5A awarded by the German government (Bundesministerium für Bildung, Wissenschaft, Forschung und Technologie) and by INTRAVIS GmbH, Herzogenrath, Germany.

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