ROBUSTNESS ANALYSIS OF MUSIC AND ESPRIT FREQUENCY ESTIMATORS FOR SINUSOIDAL SIGNALS WITH TIME-VARYING AMPLITUDE

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Abstract

In this paper, we address the problem of estimating the frequency of a sinusoidal signal with random, lowpass amplitude. We propose to use MUSIC and ESPRIT frequency estimators as if the signal had a constant amplitude. The aim of the paper is to analyze the degradation of performance induced by the aforementioned mismodelling. Unified expressions for the bias and variances of the MUSIC and ESPRIT frequency estimators are derived under the hypothesis of small bandwidth of the signal envelope. Numerical simulations illustrate the agreement between theoretical and empirical results and study the influence of the envelope bandwidth onto the frequency estimation performance.

1. PROBLEM FORMULATION

Sinusoidal signals with random or time varying amplitude show up in many applications of signal processing. Examples include estimation of train speed from radar echoes returned by the track[1], propagation of acoustic signals through the ocean[2]. This type of problem also occurs in radar with slowly-fluctuating targets[3], in dispersive communication channels[3] or in the analysis of lidar returns for weather applications [4]. More specifically, these problems amount to estimate the angular frequency ω_0 of the following model:

$$y(t) = \alpha(t).e^{j\omega_0 t}, t = 1, 2, ... \quad (1)$$

where the envelope $\alpha(t)$ in (1) is a random process. In many cases, $\alpha(t)$ is lowpass. More exactly, the cut-off frequency of the spectrum of $\alpha(t)$ is much smaller than ω_0 . In other words, (1) is a narrowband signal (but not a pure sinewave). Although a considerable amount of work has focused, for a long time, on the estimation of the parameters of constant amplitude sinusoidal signals[5], only recently, attention has been paid to signals of the form (1) (see [6] and references therein for a thorough treatment). Two main approaches can be taken to cope with this problem. The first consists of assuming a full model for the envelope and using explicitly its properties to derive estimators. This provides accurate but computationally intensive algorithms. However, in some applications, $\alpha(t)$ in (1) is known to be very lowpass and few methods exploit this fact to advantage. Our main goal herein is to develop frequency estimators which make use of the *a priori* information that $\alpha(t)$ is very lowpass to achieve computational simplicity. More exactly, we propose to estimate ω_0 in (1) by using either MUSIC[7] or ESPRIT[8] frequency estimators as if $\alpha(t)$ were constant(see [4] for a related approach). This second class of methods is computationally attractive but induces bias due to modelling errors: indeed, the larger the bandwidth of (1) (compared to ω_0), the more heavily biased are these estimators. This paper provides insights into the trade-offs associated with mismodelling. It illustrates the degradation of performance and gives answers to the following question: under which condition can one still conduct frequency estimation without taking into account the varying amplitude. We derive the bias of MUSIC and ESPRIT frequency estimators under the assumption of small envelope bandwidth. Additionally, the analysis to follow derives expressions for the asymptotic (large-sample) variances of the estimates. In the sequel, it is assumed that $\alpha(t)$ is a linear stationary process (with ARMA being a special case)

$$\alpha(t) = \sum_{k=0}^{\infty} h_k e(t-k) \quad (2)$$

where the weighting coefficients $\{h_k\}$ decrease exponentially to zero, as k increases, and $\{e(t)\}$ is a sequence of zero-mean independent and identically distributed random variables with

$$E\{e(t)e^*(s)\} = \lambda^2 \delta_{t,s}, E\{e(t)e(s)\} = \sigma \delta_{t,s} \quad (3)$$

where λ^2 is real-valued and σ is generally complex-valued. Let the covariance and autocorrelation sequences of $\{\alpha(t)\}$ be:

$$r_k = E\{\alpha^*(t)\alpha(t+k)\}, \quad \rho_k = r_k/r_0 \quad (4)$$

The assumption that $\alpha(t)$ is a very slowly varying signal can be expressed mathematically as $|1-\rho_k| \ll 1$, $k =$

$1..m$ for some $m > 1$. Let us define the sample autocorrelations of the process $\alpha(t)$:

$$\hat{\rho}_k = \frac{\sum_{t=k+1}^{N} \alpha(t)\alpha^*(t-k)}{\sum_{t=1}^{N} |\alpha(t)|^2} \tag{5}$$

The asymptotic covariance matrix of the errors $\left\{\sqrt{N}(\hat{\rho}_k - \rho_k)\right\}$ is needed in order to derive the variance of the proposed frequency estimates. Let us define:

$$\begin{aligned} \sigma_{k,p} &= \lim_{N\to\infty} NE\left\{(\hat{\rho}_k - \rho_k).(\hat{\rho}_p - \rho_p)^*\right\} && (6) \\ \upsilon_{k,p} &= \lim_{N\to\infty} NE\left\{(\hat{\rho}_k - \rho_k).(\hat{\rho}_p - \rho_p)\right\} && (7) \end{aligned}$$

Expressions for $\sigma_{k,p}$ and $\nu_{k,p}$ can be found in [9].

2. ANALYSIS OF THE FREQUENCY ESTIMATES

2.1 Analysis of MUSIC frequency estimate

Assuming that $\alpha(t)$ is nearly constant (in not too long a time interval) one can try to use MUSIC[7][10] for frequency estimation of the signal in (1), in the conventional way (i.e. as if the sinusoidal signal had a constant amplitude). First, let us denote $\mathbf{y}(t) = \left[\begin{array}{ccccc} y(t) & y(t+1) & . & . & y(t+m-1) \end{array}\right]^T$ and define, for $M = N - m + 1$:

$$\hat{\mathbf{R}} = \frac{1}{m}\sum_{t=1}^{M} \mathbf{y}(t)\mathbf{y}^H(t) / \sum_{t=1}^{M} |y(t)|^2 \tag{8}$$

With $\mathbf{a}(\omega) = \frac{1}{\sqrt{m}}.\left[1, e^{j\omega}, ..., e^{j(m-1)\omega}\right]^T$, it can be shown that

$$\begin{aligned} \hat{\mathbf{R}} &= \mathbf{a}(\omega_0)\mathbf{a}^H(\omega_0) + \bar{\mathbf{\Gamma}} \odot \left[\mathbf{a}(\omega_0)\mathbf{a}^H(\omega_0)\right] \\ &= \mathbf{a}(\omega_0)\mathbf{a}^H(\omega_0) + \mathbf{\Delta} && (9) \end{aligned}$$

where $\odot$ stands for the Hadamard matrix product and where the k, p element of $\bar{\mathbf{\Gamma}}$ is given by $\bar{\mathbf{\Gamma}}(k,p) = \tilde{\rho}_{k-1,p-1} - 1$ with

$$\tilde{\rho}_{k,p} = \frac{\sum_{t=1}^{M} \alpha(t+k)\alpha^*(t+p)}{\sum_{t=1}^{M} |\alpha(t)|^2} \tag{10}$$

Clearly, $\tilde{\rho}_{k,p}$ is close to $\hat{\rho}_{k-p}$: they only differ from one another by a term which is of order $1/N$ in probability. Hence, $\tilde{\rho}_{k,p}$ and $\hat{\rho}_{k-p}$ have the same asymptotic behavior in large-samples, which allows us to replace $\tilde{\rho}_{k,p}$ by $\hat{\rho}_{k-p}$ in the following asymptotic analysis. Furthermore, under the assumption made, $\bar{\mathbf{\Gamma}}$ is a matrix with small elements (or small norm). Hence the matrix $\hat{\mathbf{R}}$ in (9) has the structure exploited by MUSIC and other eigenanalysis-based frequency estimators: a low rank term which uniquely determines the frequency plus a full rank, small perturbation term. Let $\hat{\mathbf{s}}$ be the unit-norm principal eigenvector of $\hat{\mathbf{R}}$. Then spectral-MUSIC determines the frequency estimate as

$$\hat{\omega} = \arg\max_{\omega} f(\omega), \qquad f(\omega) = \tfrac{1}{2}\left|\mathbf{a}^H(\omega)\hat{\mathbf{s}}\right|^2 \tag{11}$$

which, evidently, yields the true value ω_0 in the limit as $\|\mathbf{\Delta}\| \to 0$. For $\|\mathbf{\Delta}\| \neq 0$, the MUSIC estimate above differs from ω_0, and the purpose of the following analysis is to establish the asymptotic bias (as $\|\mathbf{\Delta}\| \to 0$) and variance (as $\|\mathbf{\Delta}\| \to 0$ and $N \to \infty$) of $\hat{\omega}$. An asymptotic Taylor series expansion analysis[10] enables us to show that the estimation error in the MUSIC estimate $\hat{\omega}$ is (for small $\|\mathbf{\Delta}\|$) given by:

$$\hat{\omega} - \omega_0 \simeq -f'(\omega_0)/f''(\omega_0) \tag{12}$$

It can be proved that (see [9] for details):

$$f''(\omega_0) \simeq -\frac{m^2-1}{12} \tag{13}$$

$$f'(\omega_0) \simeq \operatorname{Re}\left[\mathbf{d}^H(\omega_0)\mathbf{\Delta}\mathbf{a}(\boldsymbol{\omega_0})\right] \tag{14}$$

Inserting (13) and (14) in (12) leads to:

$$\hat{\omega} - \omega_0 \simeq \frac{12.\operatorname{Re}\left[\mathbf{d}^H(\omega_0)\mathbf{\Delta}\mathbf{a}(\omega_0)\right]}{m^2-1} \tag{15}$$

Using the exact expressions for $\mathbf{a}(\omega)$ and $\mathbf{\Delta}$, it is readily verified that

$$\operatorname{Re}\left[\mathbf{d}^H(\omega_0)\mathbf{\Delta}\mathbf{a}(\omega_0)\right] = \frac{1}{m^2}\operatorname{Im}\left[\sum_{k,p=1}^{m}(k-1)\tilde{\rho}_{k-1,p-1}\right] \tag{16}$$

Hence, we finally obtain

$$\hat{\omega} - \omega_0 \simeq \frac{12.\operatorname{Im}\left[\sum_{k,p=1}^{m}(k-1)\tilde{\rho}_{k-1,p-1}\right]}{m^2(m^2-1)} \tag{17}$$

Therefore, with $C_M = \frac{12}{m^2(m^2-1)}$, the asymptotic bias of the estimate is readily derived from (17):

$$\begin{aligned} asbias(\hat{\omega}) &= C_M \sum_{k,p=1}^{m}\left\{(k-1)\operatorname{Im}\left[\rho_{k-p}\right]\right\} \\ &= C_M \sum_{n=1}^{m}\left\{(m-n)n\operatorname{Im}\left[\rho_n\right]\right\} && (18) \end{aligned}$$

From (6)-(7) and (17)-(18), along with the fact that $\left\{\operatorname{Im}[x]\right\}^2 = \frac{1}{2}\operatorname{Re}\left[xx^* - x^2\right]$, the asymptotic variance of the frequency estimate is obtained as

$$\begin{aligned} as\operatorname{var}(\hat{\omega}) &= \frac{C_M^2}{2}\operatorname{Re}\{\sum_{k,p,i,j=1}^{m}(k-1)(i-1) \\ &\quad \times\left\{\sigma_{k-p,i-j} - \upsilon_{k-p,i-j}\right\}\} && (19) \end{aligned}$$

2.2 Analysis of ESPRIT frequency estimate

In this section, we derive expressions for the mean and variance of the ESPRIT frequency estimate of the signal in (1) (see e.g. [8] for a discussion on ESPRIT). The ESPRIT estimate of ω_0 is obtained from the following equation:

$$\hat{\phi} \stackrel{\Delta}{=} \hat{\mu}.e^{j\hat{\omega}} = (\hat{\mathbf{s}}_1^H \hat{\mathbf{s}}_1)^{-1} \hat{\mathbf{s}}_1^H \hat{\mathbf{s}}_2 \tag{20}$$

where $\hat{\mathbf{s}}_1 = \mathbf{J}_1 \hat{\mathbf{s}}$, $\hat{\mathbf{s}}_2 = \mathbf{J}_2 \hat{\mathbf{s}}$ and $\mathbf{J}_1 = \begin{bmatrix} \mathbf{I}_{m-1} & \mathbf{0} \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{m-1} \end{bmatrix}$. As $\|\boldsymbol{\Delta}\|$ decreases, the vectors $\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2$ converge respectively to $\mathbf{a}_1 = \mathbf{J}_1 \mathbf{a}$ and $\mathbf{a}_2 = \mathbf{J}_2 \mathbf{a}$. Accordingly, $\hat{\phi}$ above converges to $e^{j\omega_0}$. A first-order Taylor series expansion allows us to write

$$\begin{aligned} &\hat{\mu} e^{j\hat{\omega}} - e^{j\omega_0} \simeq e^{j\omega_0} \left\{ \hat{\mu} - 1 + j(\hat{\omega} - \omega_0) \right\} \\ &\Rightarrow \hat{\omega} - \omega_0 \simeq \operatorname{Im} \left\{ (\hat{\phi} - e^{j\omega_0}) e^{-j\omega_0} \right\} \end{aligned} \tag{21}$$

It can be verified that, up to a first-order, the error $\hat{\phi} - e^{j\omega_0}$ is given by

$$\hat{\phi} - e^{j\omega_0} \simeq \boldsymbol{\eta}^H (\hat{\mathbf{s}} - \mathbf{a}(\omega_0)) \tag{22}$$

with $\boldsymbol{\eta}^H = \left(\mathbf{a}_1^H \mathbf{a}_1 \right)^{-1} \mathbf{a}_1^H \left\{ \mathbf{J}_1 - e^{j\omega_0} \mathbf{J}_2 \right\}$. Moreover, observing that $\boldsymbol{\eta}^H \mathbf{a}(\omega_0) = 0$, we have

$$\boldsymbol{\eta}^H \hat{\mathbf{R}} = \boldsymbol{\eta}^H \mathbf{a}(\omega_0) \mathbf{a}^H(\omega_0) + \boldsymbol{\eta}^H \boldsymbol{\Delta} = \boldsymbol{\eta}^H \boldsymbol{\Delta} \tag{23}$$

Under the hypothesis of small $\boldsymbol{\Delta}$, we therefore can write

$$\begin{aligned} \hat{\phi} - e^{j\omega_0} &\simeq \boldsymbol{\eta}^H \hat{\mathbf{s}} \simeq \boldsymbol{\eta}^H \hat{\mathbf{R}} \hat{\mathbf{s}} \\ &= \boldsymbol{\eta}^H \boldsymbol{\Delta} \hat{\mathbf{s}} \\ &\simeq \boldsymbol{\eta}^H \boldsymbol{\Delta} \mathbf{a}(\omega_0) \end{aligned} \tag{24}$$

where to obtain the second equality we made use of the fact that the principal eigenvalue of $\hat{\mathbf{R}}$, associated with $\hat{\mathbf{s}}$, goes to one as $\|\boldsymbol{\Delta}\| \to 0$. Inserting the previous equation in (21) leads to the following expression for the error of the ESPRIT frequency estimate:

$$\hat{\omega} - \omega_0 \simeq \operatorname{Im} \left\{ \boldsymbol{\beta}^H \boldsymbol{\Delta} \mathbf{a}(\omega_0) \right\} \tag{25}$$

with $\boldsymbol{\beta} = e^{j\omega_0} \boldsymbol{\eta} = \frac{\sqrt{m}}{m-1} \left[-1, 0, ..., 0, e^{j(m-1)\omega_0} \right]^T$. By using the expressions of $\boldsymbol{\eta}, \mathbf{a}_1, \mathbf{a}_2$, one can show that

$$\hat{\omega} - \omega_0 \simeq \frac{1}{m(m-1)} \operatorname{Im} \left\{ \sum_{p=1}^{m} \left[\tilde{\rho}_{m-1,p-1} - \tilde{\rho}_{0,p-1} \right] \right\} \tag{26}$$

Therefore, with $C_E = \frac{1}{m(m-1)}$, the asymptotic bias of ESPRIT estimate is given by:

$$\begin{aligned} asbias\left(\hat{\omega} \right) &\simeq C_E \operatorname{Im} \left\{ \sum_{p=1}^{m} \left[\rho_{m-p} - \rho_{1-p} \right] \right\} \\ &= 2 C_E \sum_{p=1}^{m} \operatorname{Im} \left[\rho_{m-p} \right] \end{aligned} \tag{27}$$

From (26), we also obtain the asymptotic variance of the ESPRIT frequency estimate

$$\begin{aligned} as\operatorname{var}(\hat{\omega}) = \operatorname{Re}\{ \frac{C_E^2}{2} \sum_{p,n=1}^{m} & \sigma_{m-p,m-n} - \sigma_{1-p,m-n} \\ & + \sigma_{1-p,1-n} - \sigma_{m-p,1-n} \\ & - \upsilon_{m-p,m-n} - \upsilon_{1-p,m-n} \\ & + \upsilon_{1-p,1-n} - \upsilon_{m-p,1-n} \} \end{aligned} \tag{28}$$

In closing this section, we note that the previous bias analysis results can be presented in a unified manner as follows. Defining $\boldsymbol{\gamma} = [\operatorname{Im}(\rho_1), ..., \operatorname{Im}(\rho_{m-1})]^T$, $\boldsymbol{\alpha} = [\alpha_1, ..., \alpha_m]^T$, the bias expressions (18) and (27) can be rewritten in the form

$$asbias(\hat{\omega}) = \boldsymbol{\alpha}^T \boldsymbol{\gamma} \tag{29}$$

where $\alpha_k = \frac{12k(m-k)}{m^2(m^2-1)}$ for MUSIC and $\alpha_k = \frac{2}{m(m-1)}$ for ESPRIT. In particular, (29) implies that the estimates are unbiased if the correlation sequence $\{\rho_k\}$ is real-valued.

3. NUMERICAL ILLUSTRATIONS

We now illustrate the validity of the previous analysis and the respective performances of the MUSIC and ESPRIT estimators. The envelope is chosen as a complex $AR(2)$ process with poles $\rho_1.e^{j2\pi f_1}$, $\rho_2.e^{j2\pi f_2}$. The frequency ω_0 was equal to $2\pi 0.18$. The number of samples was selected as $N = 400$ and the value of m was fixed to $m = 10$. Two cases are considered:

case 1: $\rho_1 = \rho_2 = 0.98$, $f_1 = -0.006$;

case 2: $\rho_1 = 0.98, \rho_2 = 0.95$, $f_1 = -0.006$.

f_2 was varied between 0 and 0.01. 200 Monte Carlo simulations were run to estimate the MSE of the estimators. It should be noted that, in case 1, the envelope can be real-valued (*e.g.* when $f_2 = -f_1$) whereas this is no longer possible in case 2. Figures 1-2 show the MSE's for ESPRIT and MUSIC for varying f_2.

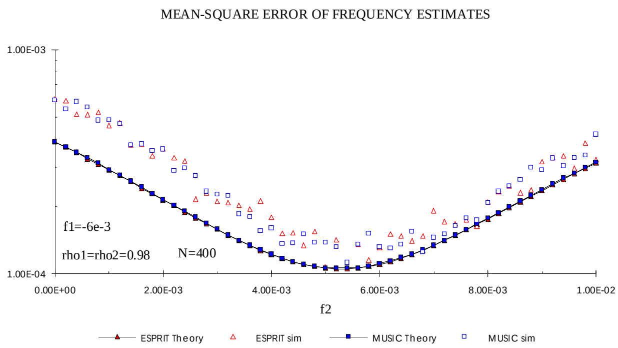

Fig. 1: MSE of MUSIC and ESPRIT frequency estimates versus f_2. $N = 400$. $m = 10$. $\rho_1 = 0.98$. $\rho_2 = 0.98$. $f_1 = -0.006$.

As can be seen from these figures, the empirical results are in good agreement with the theory. Also note the different forms of the curves from case 1 to case 2. In the first case, when $f_2 \approx -f_1$ the estimates are unbiased (hence the MSE achieves a minimum): this corresponds to a real-valued envelope correlation. It should be pointed out that the squared bias term is larger than the variance when f_2 is not too close to $-f_1$. In case 2, it could be observed (see [9]) that the squared bias is more severe than in case 1 and always constitutes the predominant term of the MSE. This is due to the fact that the envelope varies faster and that the envelope correlation can no longer be real-valued.

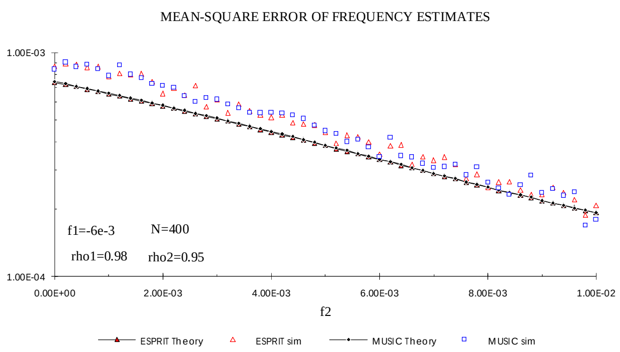

Fig. 2: MSE of MUSIC and ESPRIT frequency estimates versus f_2. $N = 400$. $m = 10$. $\rho_1 = 0.98$. $\rho_2 = 0.95$. $f_1 = -0.006$.

Next, we study the influence of m on the MSE's of ESPRIT and MUSIC. The envelope parameters are now $\rho_1 = 0.98, \rho_2 = 0.95, f_1 = -0.006, f_2 = 0.004$. The number of data samples is $N = 400$ and m is varied between 2 and 30. Figure 3 displays the MSE's of the estimates.

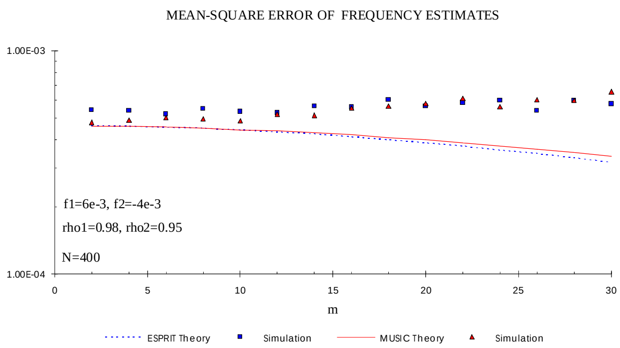

Fig. 3: MSE of MUSIC and ESPRIT frequency estimates versus m. $N = 400$. $\rho_1 = 0.98$. $\rho_2 = 0.95$. $f_1 = -0.006$. $f_2 = 0.004$.

It can be seen that increasing m does not significantly improve the theoretical accuracy of estimation, in contrast to what has been noticed in the constant amplitude case. In fact, we observed[9] that the empirical bias increases with m whereas the variance slightly decreases. Also, the agreement between theoretical and empirical results becomes poorer as m increases. This is due to the fact that, as m increases, the assumption of a constant envelope over m samples is less and less valid. Consequently, the theoretical analysis does not predict well the empirical results, for large m.

4. CONCLUSIONS

In this paper, the robustness of MUSIC and ESPRIT frequency estimators for sinusoidal signals with random, time-varying amplitude was analyzed. Using these estimators as if the amplitude was constant results in a bias for both estimators. Under the assumption of a slowly-varying envelope, we derived unified expressions for these biases and showed their relation with the envelope correlation. It was observed that MUSIC and ESPRIT have practically the same performance (a similar result had already been shwon for constant amplitude signals [10]). Finally, it was pointed out that increasing the size m of the correlation matrix does not result in an improvement of accuracy.

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