PERFORMANCE ANALYSIS OF A WAVELET BASED WBCAF METHOD FOR TIME DELAY AND DOPPLER STRETCH ESTIMATION

X. X. Niu* P. C. Ching* Y. T. Chan+

*Dept. of Electronic Engineering, Chinese University of Hong Kong, Hong Kong

+Dept. of Electrical Engineering, Royal Military College of Canada, Canada

Tel: (852) 2609 8275 Fax: (852) 2603 5558

Email: pcching@ee.cuhk.edu.hk

ABSTRACT

A wavelet based method for time delay and Doppler stretch estimation has been proposed. It makes use of the relationship between the wideband cross ambiguity function (WBCAF) and the cross wavelet transform of the received signals. This paper derives the Cramer-Rao lower bound (CRLB) and analyses the performance of the algorithm. It is found that under high SNR, the method is asymptotically unbiased, and the variances of the estimation parameters are fairly close to the CRLB. Simulation results are given to corroborate the theoretical derivation.

1 INTRODUCTION

In radar, sonar or global positioning systems (GPS), the time delay and Doppler stretch are two essential parameters for localization of the target position and to measure its movement. The Doppler stretch is generally defined as the time scaling of a wideband signal. Jin *et al* [1] have analysed the performance of joint estimation of time delay and Doppler stretch by using the wideband cross ambiguity function (WBCAF). The method is applied to the case of active radar in which one of the two sensors is assumed to be noise free. This algorithm, however, is not applicable in the situation when the two sensors are both corrupted by additive noise. Recently, we proposed a wavelet based WBCAF (WB-WBCAF) algorithm to estimate the scale and time delay using cross wavelet transform in a more generalized form [2]. The method requires more computation but has wider range of applications. In this paper, we derive the Cramer-Rao lower bound (CRLB) and analyse the performance of this WB-WBCAF method. Simulation results are also provided to illustrate the potential of the method.

2 WB-WBCAF JOINT ESTIMATION METHOD

In passive radar or GPS systems, the source signal is usually submerged in contaminated noise and is mostly unknown. It is, therefore, very difficult to calculate the WBCAF between two received signals directly because it is almost impossible to obtain an arbitrary scaled and delayed version of a data sequence. Weiss [3] and Young [4] have introduced the relationship between WBCAF and the wavelet transform. By making use of this finding we recently proposed a wavelet based WBCAF method that provides simultaneous estimation of both the time delay and Doppler stretch [2].

Suppose the two received signals are given by

$$\begin{cases} f_1(t) &= s(t) + n_1(t) \\ f_2(t) &= s_r(t) + n_2(t) \end{cases} \tag{1}$$

where $s(t)$ is the source signal received by one sensor and $s_r(t)$ is the signal received by the other sensor that incorporates an initial time delay and a scale factor caused by the target movement, i.e. $s_r(t) = s(\frac{t-\tau_0}{\sigma_0})$. Whilst $n_1(t)$ and $n_2(t)$ are ergodic, white, zero-mean Gaussian processes with the same power spectral density, N_0, and they are assumed to be uncorrelated to each other. The cross wavelet transform [4] of two arbitrary signals, $f_1(t)$ and $f_2(t)$, is given by

$$A_{f_2 f_1}(\tau,\sigma) = \frac{1}{c_h}\int\int_{-\infty}^{\infty} W_h f_2(a,b)\, W_h^* f_1(\frac{a}{\sigma}, \frac{b-\tau}{\sigma})\frac{da\,db}{a^2} \tag{2}$$

where $W_h f_i$ denotes the wavelet transform of $f_i(t)$ ($i = 1,2$) with respect to a particular mother wavelet, $h(t)$, which can be computed from

$$W_h f_i(a,b) \triangleq \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} f_i(t)\, h^*(\frac{t-b}{a})\, dt \qquad i = 1,2 \tag{3}$$

The parameter c_h in (2) is the admissible constant of the wavelet $h(t)$. Now the WBCAF of the two received signals can be equated to their cross wavelet transform, that is

$$\begin{aligned} \mathrm{WBCAF}(\tau,\sigma) &\triangleq \frac{1}{\sqrt{|\sigma|}}\int_{-\infty}^{\infty} f_2(t)\, f_1^*\left(\frac{t-\tau}{\sigma}\right) dt \qquad (4) \\ &= A_{f_2 f_1}(\tau,\sigma) \\ &= \frac{1}{c_h}\int\int_{-\infty}^{\infty} W_h f_2(a,b)\, W_h^* f_1(\frac{a}{\sigma}, \frac{b-\tau}{\sigma})\, \frac{da\,db}{a^2} \end{aligned}$$

In this application, it is necessary to find a mother wavelet $h(t)$ which is analytic so that $h(\frac{t-\frac{b-\tau}{\sigma}}{\frac{a}{\sigma}})$ can be obtained for different values of σ and τ in advance. The maximum point of the $WBCAF(\tau,\sigma)$ is then located which is uniquely mapped to the required estimation of τ_0 and σ_0 at the (τ,σ) plane.

3 THE CRAMER-RAO LOWER BOUND

The CRLB is the lowest theoretical bound for the variance of unbiased estimators which can be obtained by using the maximum likelihood (ML) estimation [5]. The parameter vector to be estimated in (1) is $\mathbf{g} = [\tau_0 \;\; \sigma_0]^T$. The references of τ_0 and σ_0 are 0 and 1 respectively, but these reference points will also be affected by the corrupting noise $n_1(t)$. So we rewrite (1) as follows

$$\begin{cases} f_1(t) = s(\frac{t-\tau_1}{\sigma_1}) + n_1(t) \\ f_2(t) = s(\frac{t-\tau_2}{\sigma_2}) + n_2(t) \end{cases} \tag{5}$$

where $\frac{\sigma_2}{\sigma_1} = \sigma_0$, $\frac{\tau_2-\tau_1}{\sigma_1} = \tau_0$ and $E[\sigma_1] = 1$, $E[\tau_1] = 0$. The parameters to be estimated now become τ_1, σ_1, τ_2 and σ_2, which can be denoted by $\theta = [\tau_1 \;\; \sigma_1 \;\; \tau_2 \;\; \sigma_2]^T$.

From the estimation theory [5], the covariance matrix of $\hat{\theta}$ is given by the inequality

$$\mathrm{cov}(\hat{\theta}) \geq \mathbf{I}^{-1}(\theta) \tag{6}$$

and the ii-th element of the covariance matrix represents the CRLB of the element. $\mathbf{I}(\theta)$ is the $p \times p$ Fisher information matrix which is defined as

$$\begin{aligned} [\mathbf{I}(\theta)]_{ij} &= -E\left[\frac{\partial^2 \ln p(f_1,f_2;\theta)}{\partial\theta_i \partial\theta_j}\right] \\ &= E\left[\frac{\partial \ln p(f_1,f_2;\theta)}{\partial\theta_i}\frac{\partial \ln p(f_1,f_2;\theta)}{\partial\theta_j}\right] \end{aligned} \tag{7}$$

The conditional probability density function (PDF) of the received data can be expressed as

$$\begin{aligned} & p(f_1,f_2;\theta) = p(f_1;\theta)p(f_2;\theta) \\ = & K_1 \exp\left\{-\frac{1}{N_0}\int\left|f_1(t) - s\left(\frac{t-\tau_1}{\sigma_1}\right)\right|^2 dt\right\} \\ \cdot & K_2 \exp\left\{-\frac{1}{N_0}\int\left|f_2(t) - s\left(\frac{t-\tau_2}{\sigma_2}\right)\right|^2 dt\right\} \end{aligned} \tag{8}$$

where K_1 and K_2 are constants.

The covariance matrix of $\hat{\theta}$ is derived by using (6), (7) and (8), and has the following form

$$cov(\hat{\theta}) \geq \frac{N_0}{2\Delta_b(s)}\begin{bmatrix} \sigma_1 D_s & -\sigma_1 C_s & 0 & 0 \\ -\sigma_1 C_s & \sigma_1 B_s & 0 & 0 \\ 0 & 0 & \sigma_2 D_s & -\sigma_2 C_s \\ 0 & 0 & -\sigma_2 C_s & \sigma_2 B_s \end{bmatrix} \tag{9}$$

where $B_s \triangleq \int_{-\infty}^{\infty} |\dot{s}(t)|^2 dt$, $C_s \triangleq \int_{-\infty}^{\infty} t|\dot{s}(t)|^2 dt$, $D_s \triangleq \int_{-\infty}^{\infty} t^2|\dot{s}(t)|^2 dt$ and $E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt$, and $\Delta_b(s) = D_s B_s - C_s^2$.

The covariance matrix of $\mathbf{g}$, which is a function of θ, can be obtained by using the transformation of parameters in estimation theory [5]

$$cov(\hat{\mathbf{g}}) \geq \frac{\partial \mathbf{g}}{\partial\theta} cov(\hat{\theta}) \frac{\partial \mathbf{g}}{\partial\theta}^T \tag{10}$$

The CRLB of $\hat{\tau}_0$ and $\hat{\sigma}_0$ corresponding to the ii-th elements of matrix $cov(\hat{\mathbf{g}})$ is given by

$$\begin{aligned} var(\hat{\tau}_0)_{CRLB} &= \frac{N_0}{2\Delta_b(s)}[(1+\sigma_0)D_s + \tau_0^2 B_s - 2\tau_0 C_s] \\ var(\hat{\sigma}_0)_{CRLB} &= \frac{N_0}{2\Delta_b(s)}\sigma_0(1+\sigma_0)B_s \end{aligned} \tag{11}$$

This CRLB is about twice as large as that of the active radar case [1]. It is reasonable because, in the passive case, the reference signal is also corrupted by noise, hence the estimation would be less accurate.

4 PERFORMANCE ANALYSIS OF THE WB-WBCAF

In this section, the theoretical variances of τ_0 and σ_0 estimated by the proposed method are given and compared with the CRLB.

The WB-WBCAF consists of four parts according to (2) and (4):

$$\begin{aligned} WBCAF(\tau,\sigma) = \;& A_{s_r s}(\tau,\sigma) + A_{n_2 s}(\tau,\sigma) \\ & + A_{s_r n_1}(\tau,\sigma) + A_{n_2 n_1}(\tau,\sigma) \end{aligned} \tag{12}$$

The estimation of τ and σ requires locating the maximum point of $|WBCAF(\tau,\sigma)|^2$, (τ_p,σ_p), which can be determined by

$$\begin{aligned} 0 &= \tfrac{\partial}{\partial\tau_p}\left|WBCAF(\tau_p,\sigma_p)\right|^2 \\ &\approx 2\sqrt{\sigma_0}E_s Re\left\{\tfrac{\partial}{\partial\tau_p}WBCAF^*(\tau_p,\sigma_p)\right\} \quad prob. \\ 0 &= \tfrac{\partial}{\partial\sigma_p}\left|WBCAF(\tau_p,\sigma_p)\right|^2 \\ &\approx 2\sqrt{\sigma_0}E_s Re\left\{\tfrac{\partial}{\partial\sigma_p}WBCAF^*(\tau_p,\sigma_p)\right\} \quad prob. \end{aligned} \tag{13}$$

Expanding $Re\left[A_{s_r s}(\tau,\sigma)\right]$ in Talor series around (τ_0,σ_0), and ignoring higher order terms, we have

$$\begin{aligned} Re\left[A_{s_r s}(\tau,\sigma)\right] \approx\; & Re\left[A_{s_r s}(\tau_0,\sigma_0)\right] + \tfrac{1}{2}V_{11}(\tau-\tau_0)^2 \\ & + V_{12}(\tau-\tau_0)(\sigma-\sigma_0) + \tfrac{1}{2}V_{22}(\sigma-\sigma_0)^2 \end{aligned} \tag{14}$$

where V_{11}, V_{12} and V_{22} are some constants related to the source signal as defined in [1]. Taking partial differentiations of (14) with respect to τ and σ and using (12) and (13), we get

$$\begin{aligned} \hat{\tau}_0 \approx\; & \tau_0 + \frac{\sigma_0^3}{\Delta(s)}\Big\{V_{12}\,Re\left[A'_{n_2 s}(\tau_0,\sigma_0)\right] \\ + \; & V_{12}\,Re\left[A'_{s_r n_1}(\tau_0,\sigma_0)\right] + V_{12}\,Re\left[A'_{n_2 n_1}(\tau_0,\sigma_0)\right] \\ - \; & V_{22}\,Re\left[\dot{A}_{n_2 s}(\tau_0,\sigma_0)\right] - V_{22}\,Re\left[\dot{A}_{s_r n_1}(\tau_0,\sigma_0)\right] \end{aligned}$$

$$
\begin{aligned}
&- \quad V_{22}\, Re\left[\dot{A}_{n_2 n_1}(\tau_0,\sigma_0)\right]\Big\} \\
\hat{\sigma}_0 &\approx \sigma_0 + \frac{\sigma_0^3}{\Delta(s)}\Big\{V_{12}\, Re\left[\dot{A}_{n_2 s}(\tau_0,\sigma_0)\right] \\
&+ \quad V_{12}\, Re\left[\dot{A}_{s_r n_1}(\tau_0,\sigma_0)\right] + V_{12}\, Re\left[\dot{A}_{n_2 n_1}(\tau_0,\sigma_0)\right] \\
&- \quad V_{11}\, Re\left[A'_{n_2 s}(\tau_0,\sigma_0)\right] - V_{11}\, Re\left[A'_{s_r n_1}(\tau_0,\sigma_0)\right] \\
&- \quad V_{11}\, Re\left[A'_{n_2 n_1}(\tau_0,\sigma_0)\right]\Big\} \qquad (15)
\end{aligned}
$$

where $\Delta(s) = B_s(D_s - \frac{E_s}{4}) - C_s^2$, $\dot{A} = \partial A/\partial\tau$ and $A' = \partial A/\partial\sigma$. $A_{xy}(\tau,\sigma)$ again denotes the cross wavelet transform of $x(t)$ and $y(t)$ with respect to a mother wavelet $h(t)$, as defined in (2).

Taking expectation on (15) and using the following equalities:

$$
\begin{array}{ll}
E[n_i(t)] = 0 & i = 1,2 \\
E[n_i(t_1)n_j(t_2)] = 0 & i,j = 1,2 \\
E[n_i(t_1)n_i^*(t_2)] = N_0\delta(t_1 - t_2) & i = 1,2 \\
E[n_1(t_1)n_2^*(t_2)] = 0 &
\end{array} \qquad (16)
$$

the estimation can be shown to be asymptotically unbiased when SNR goes to infinity, in this case

$$
\begin{aligned}
E[\hat{\tau}_0] &\approx \tau_0 \qquad (17) \\
E[\hat{\sigma}_0] &\approx \sigma_0
\end{aligned}
$$

From (15), by going through the tedious but straightforward derivation, we obtain the variances of the estimation

$$
\begin{aligned}
var(\hat{\tau}_0) &= \frac{N_0\sigma_0(1+\sigma_0)}{2\Delta(s)}\left(D_s - \frac{E_s}{4}\right) \qquad (18) \\
&+ \frac{N_0^2\sigma_0^6}{2c_h\Delta^2(s)}\left[V_{12}^2\frac{X - U/4}{\sigma_0^2} + V_{22}^2 V - 2V_{12}V_{22}\frac{W}{\sigma_0}\right] \\
var(\hat{\sigma}_0) &= \frac{N_0\sigma_0(1+\sigma_0)}{2\Delta(s)}B_s \\
&+ \frac{N_0^2\sigma_0^6}{2c_h\Delta^2(s)}\left[V_{12}^2 V + V_{11}^2\frac{X - U/4}{\sigma_0^2} - 2V_{12}V_{11}\frac{W}{\sigma_0}\right]
\end{aligned}
$$

The first terms of (18) are fairly similar to the CRLB (11), while the second terms have components due in part to the effect of the mother wavelet in WB-WBCAF. U, V, W and X are some constants of the mother wavelet, $h(t)$, which are given by

$$
\begin{aligned}
U &\triangleq \iint_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(t - \frac{b-\tau_0}{a})|^2\, dt\, \frac{db\, da}{a^2} \\
V &\triangleq \iint_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{h}(t - \frac{b-\tau_0}{a})|^2\, \frac{dt}{a^2}\, \frac{db\, da}{a^2} \\
W &\triangleq \iint_{-\infty}^{\infty}\int_{-\infty}^{\infty} t|\dot{h}(t - \frac{b-\tau_0}{a})|^2\, \frac{dt}{a}\, \frac{db\, da}{a^2} \\
X &\triangleq \iint_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^2|\dot{h}(t - \frac{b-\tau_0}{a})|^2\, dt\, \frac{db\, da}{a^2}
\end{aligned} \qquad (19)
$$

These constants can be simplified by using the properties of the mother wavelet,

$$
\begin{aligned}
U &\approx \frac{2TE_h}{a_i} \qquad & V &\approx \frac{2TB_h}{3a_i^3} \\
W &\approx 0 \qquad & X &\approx \frac{2TD_h}{a_i} + \frac{B_hT^3}{18a_i^3}
\end{aligned} \qquad (20)
$$

where B_h, D_h and E_h have the same definitions as B_s, D_s and E_s, while T denotes the time duration of the source signal. The parameter a_i is the minimum scale value in which the scaled wavelet, $h(\frac{t}{a_i})$, still satisfies the conditions of being a mother wavelet. If $|a| < |a_i|$, the scaled version, $h(\frac{t}{a})$, will no longer be a wavelet due to too much compression on $h(t)$.

If the source signal and the mother wavelet are all symmetric functions, the constants $C_s \approx 0$ and $V_{12} \approx 0$. The theoretical variances of the estimation can then be simplified as

$$
\begin{aligned}
var(\hat{\tau}_0) &\approx \frac{N_0\sigma_0(1+\sigma_0)}{2B_s} + \frac{N_0^2}{3c_h}\left(\frac{\sigma_0}{a_i}\right)^3\frac{T\,B_h}{B_s^2} \qquad (21) \\
var(\hat{\sigma}_0) &\approx \frac{N_0\sigma_0(1+\sigma_0)}{2(D_s - E_s/4)} \\
&+ \frac{N_0^2\sigma_0}{2c_h(D_s - E_s/4)^2}\left[\frac{2T}{a_i}(D_h - \frac{E_h}{4}) + \frac{B_hT^3}{18a_i^3}\right]
\end{aligned}
$$

The first terms of $var(\hat{\tau}_0)$ and $var(\hat{\sigma}_0)$ are very close to their CRLBs which will be shown numerically in the next section whereas the second terms depend on the noise spectral density, the source signal and the mother wavelet.

5 SIMULATION RESULT AND DISCUSSION

In our simulation, the signal is chosen to be a linear FM signal, $s(t) = 0.475\cos(0.34\pi t^2)[u(t+4) - u(t-4)]$. The mother wavelet is a Gaussian windowed sinusoidal signal, $h(t) = 1.2472e^{-0.95t^2}\cos(2\pi 1.5t)$. For this particular source signal and mother wavelet, the constants including B_s, C_s, D_s, E_s, U, V, W and X can be calculated explicitly and the CRLB together with the performance analysis are plotted. Fig.1(a) shows the variance of the time delay and Fig.1(b) depicts the variance of the scale in dB respectively. The solid line shows the CRLB derived in (11). The dotted line (very close to the solid line in (a) and exactly identical to in (b)) shows the first terms of (21) which depend only on the source signal and the noise strength. The dash-dotted line gives the theoretical variance of the estimation from the performance analysis of our WB-WBCAF method, viz (21). The discrepancy between the theoretical variance and the CRLB is caused by the second terms of (21) which are affected by the noise power, the choice of mother wavelet and source signal. The large deviation of the theoretical variance from the CRLB in low SNR is probably due to the fact that in our derivation the assumption of high

SNR is used. So the theoretical variance under noisy environment only gives an approximation. The simulation result of the proposed algorithm is marked by the asterisks. In low SNR the effect of noise is so severe that the estimation has a fairly large variance compared with the CRLB. In high SNR, the accuracy of estimation is much improved and, of course, is still not as good as the CRLB. This is due to the choice of the source signal because an optimum source signal is generally required.

6 CONCLUSION

The WB-WBCAF method for joint time delay and Doppler stretch estimation has been proved to be asymptotically unbiased when the signal-to-noise ratio approaches to infinity. The theoretical variances of the proposed WB-WBCAF estimator are very close to the CRLB in high SNR. In low SNR condition, the accuracy of estimation is significantly affected by the additive noise. A technique called wavelet denoising is introduced as a front end process and the effect of noise is shown to be greatly reduced. Further investigation in this area is still undergoing.

References

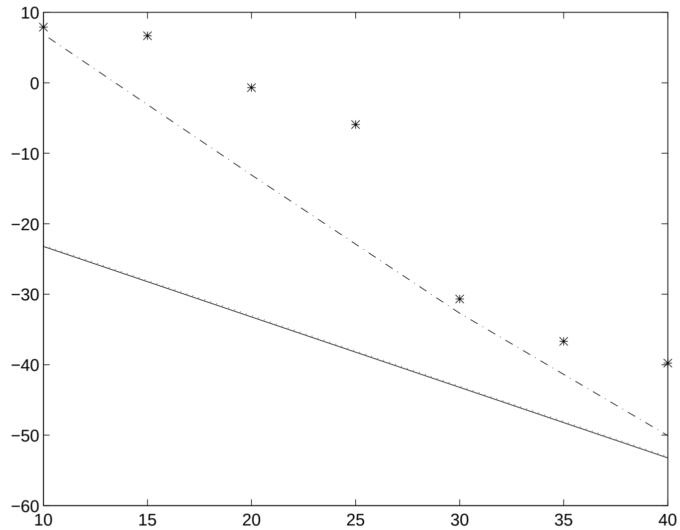
[1] Q. Jin, K. M. Wong and Z. Q. Luo, "The estimation of time delay and doppler stretch of wideband signals", *IEEE Trans. on Signal Processing*, vol. SP-43, no.4, pp.904-916, 1995.

[2] X. X. Niu, P. C. Ching and Y. T. Chan, "A wavelet-based algorithm for time delay and doppler measurements", *International Symposium on Signal, System and Electronics, San Francisco, 1995.*

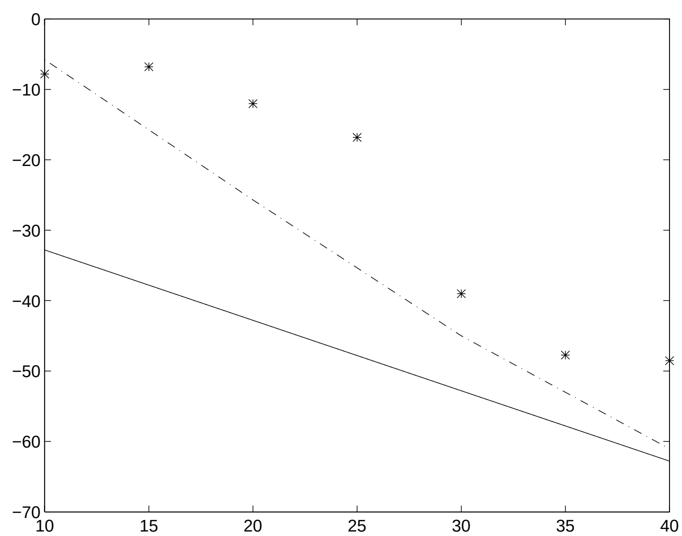
[3] L. G. Weiss, "Wavelets and wideband correlation processing", *IEEE Signal Processing Magazine*, Jan. pp.13-32, 1994.

[4] R. K. Young, *Wavelet theory and its applications*, Kluwer Academic Publishers, Boston, 1993.

[5] Steven M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, PTR Prentice Hall, Englewood Cliffs, New Jersey, 1993.



(a)



(b)

Figure 1: Comparison of the CRLB (solid line), the theoretical variance of the WB-WBCAF method (dash-dotted line) and the simulation results of WB-WBCAF method (asterisks) for (a) time delay, and (b) scale estimation.