NONLINEAR DYNAMICS OF BANDPASS SIGMA-DELTA MODULATION

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ABSTRACT

Much research attention in recent years has been focussed on the subject of oversampled analogue-to-digital and digital-to-analogue conversion, based on the principle of sigma-delta modulation. Theoretical analysis of these conversion methods has been complicated by their nonlinear nature, precluding the application of standard linear circuit analysis methods. In recent years a number of researchers have undertaken a study of sigma-delta modulation based on nonlinear methods. This paper summarises the results that have been obtained by this study in the case of bandpass sigma-delta modulation, and shows how these results can be extended to handle certain circuit nonidealities.

1. INTRODUCTION

Sigma-Delta (ΣΔ, also known as Delta-Sigma) modulation has become in recent years an increasingly popular choice for robust and inexpensive analogue-to-digital and digital-to-analogue conversion. This modulation scheme relaxes demands on amplitude resolution at the expense of increased sampling rate, and relaxes demands on analogue circuitry at the expense of increased demands on digital circuitry. Both of these trade-offs are well suited to the characteristics of today's technology. Most applications of this process in data conversion require that the signal be sampled at a rate substantially greater than the normal Nyquist rate, giving rise to the use of the term "oversampled" to describe these converters.

The history of ΣΔ modulation can be traced to a number of sources, a disparity reflected in part by the use in the literature of the names "Sigma-Delta" and "Delta-Sigma" to describe the same technique. Most researchers now credit Cutler's 1960 patent [1] with the first published account of the ΣΔ principle, albeit not in the form used throughout the modern literature. Inose, Yasuda and Murakami [2,3] extended this work in the early 1960s, introducing the name "Delta-Sigma" and providing much of the original analysis of the technique. Candy and his colleagues, in a series of papers beginning in 1974 [4-6], introduced the name "Sigma-Delta" and popularised the use of the technique for analogue-to-digital conversion, particularly in an integrated-circuit environment. The explosion of research interest in these systems since then is documented in the IEEE reprint volume [7], and many good tutorial overviews of the subject have been written [8].

Despite the popularity of these converters, understanding of the operation of ΣΔ systems is far from complete. This is due to the nonlinear nature of the modulation process. The response of most engineers confronted with a nonlinear system is to approximate it by a linear system to which standard linear analysis techniques can be applied. This "linearise, then analyse" approach pervades much of the scientific literature on ΣΔ modulation. Many of the results thus obtained are incorrect not only quantitatively, but also qualitatively. This inadequacy of linear approximation techniques when applied to ΣΔ systems has led some researchers during recent years to begin to develop a theory of ΣΔ modulation which takes account of the intrinsic nonlinearity. Methods used have included those of ergodic theory, geometry, and – the approach used in this paper – nonlinear dynamics.

ΣΔ modulators operate in discrete-time, and convert a time-sampled analogue input signal to a stream of bits. The coarse (one-bit) nature of the output of these converters requires that the system operate at a frequency many times greater than the highest frequency of interest in the input, with the output bits subsequently averaged (in some sense) to retrieve a good approximation to the input. Thus standard ΣΔ modulators operate successfully on low-frequency signals. A variation on the standard topology yields bandpass modulators, which operate on higher frequency narrowband signals. Bandpass ΣΔ modulators [9,10] can achieve high accuracy at high frequencies for a bandlimited input, and have applications in areas such as digital radio demodulation.

Most analyses of bandpass ΣΔ modulation that have appeared in the literature to date have employed linearising assumptions, despite the fact that the operation of the process depends in a crucial way on the presence of a nonlinear element – a one-bit quantiser. This paper outlines the results that can be attained by application of nonlinear dynamics to these systems, and discusses the effect of certain circuit nonidealities on these systems.

2. BANDPASS ΣΔ MODULATION

The basic structure of a bandpass ΣΔ modulator is as shown in Figure 1, and consists of a discrete-time filter followed by a one-bit quantiser in a feedback loop.

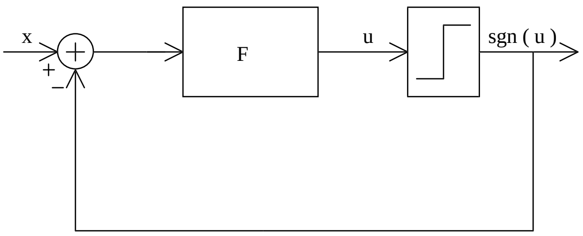

Figure 1. Bandpass ΣΔ modulator

The standard analysis of these systems involves replacing the quantiser by an additive white noise source uncorrelated with the input, thereby allowing the application of linear theory. Based on this linear analysis, the transfer function from noise source to output is easily seen to be $\frac{1}{1+F}$, while the transfer function from input to output is $\frac{F}{1+F}$. In a bandpass ΣΔ modulator the filter transfer function F is chosen so that the noise transfer function is zero and the signal transfer function unity at a frequency f_0, which will be the centre frequency of the passband of the modulator. The modulator output (a stream of bits) can then be bandpass filtered to remove the out-of-band quantisation noise and retrieve a good approximation to the input signal.

We consider first the second-order bandpass ΣΔ modulator with noise transfer function zeroes at $e^{\pm j\theta}$, where θ is chosen to centre the passband at any required frequency $f_0 = f_s.\theta/2\pi$. (f_s is the sampling frequency.) The noise transfer function is, therefore, $1 - 2\cos\theta.z^{-1} + z^{-2}$, which corresponds to a filter

$$F(z)=\frac{2\cos\theta.z^{-1}-z^{-2}}{1-2\cos\theta.z^{-1}+z^{-2}} \qquad (1)$$

Linear analysis of bandpass ΣΔ modulators based on the approach outlined above, while yielding some intuition, cannot explain the rich dynamical behaviour of these systems. For a full analysis it is necessary to take account of the nonlinearity of the system. The system of Figure 1, with F given by (1), is modelled by the difference equation

$$u_n = 2\cos\theta.u_{n-1} - u_{n-2} + 2\cos\theta.(x_{n-1} - sgn(u_{n-1})) - (x_{n-2} - sgn(u_{n-2})) \qquad (2)$$

Alternative bandpass topologies give rise to different models. The dynamics of such equations with zero and sinusoidal input were described in Reference 11 and are summarised here. The system is described by a second-order difference equation, and so the state space is again two-dimensional. We will consider the behaviour of the trajectories of (2) in the $u_n u_{n+1}$ plane. With zero input, the fixed and periodic points of (2) can be found by a matrix method. The theory of nonlinear difference equations can be used to prove that initial conditions near these periodic points give rise to trajectories that move elliptically around the periodic points. Other initial conditions can give rise to more complex behaviour, tracing out a fractal pattern involving infinitely many ellipses, as shown in Figure 2.

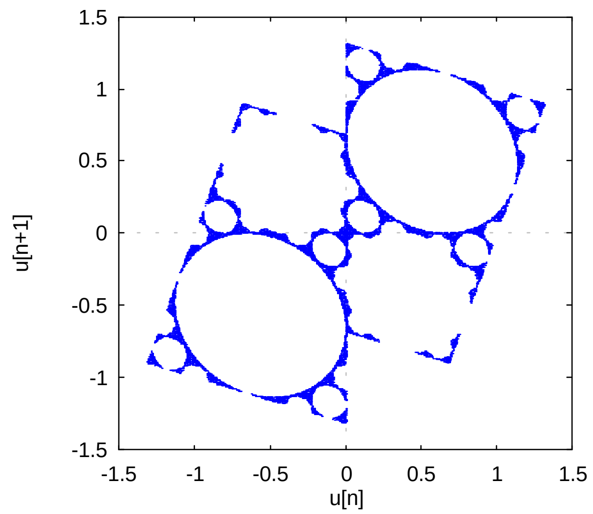

Figure 2. Trajectory of the zero-input bandpass ΣΔ system modelled by (2) with $2\cos\theta$ = -.317.

The trapping region in which the trajectories are confined can easily be determined, giving the bounds on the integrator outputs. The elliptical neighbourhoods, some of which can be seen in the figure, are regions about the periodic points in which all initial conditions give rise to periodic output sequences. This can give rise to an effect similar to the creation of idle tones in lowpass ΣΔ modulators, where zero or dc inputs can give rise to periodic output sequences which, if they lie within the signal band, appear as unwanted tones in the filtered output.

Readers familiar with the work of Chua and Lin [12] on complex behaviour in digital filters will already have recognised the similarity between the behaviour of this bandpass ΣΔ system and that of the digital filter with two's complement accumulator overflow. The applicability of the work of Chua and Lin to the bandpass ΣΔ modulator with zero input is another example of the universal nature of the behaviour often observed in families of nonlinear systems.

The case of sinusoidal input is more complicated, but can gain be analysed using techniques similar to those which have been applied to the zero-input system. Instead of elliptical trajectories around periodic points, the system exhibits toral trajectories, such as that of Figure 3.

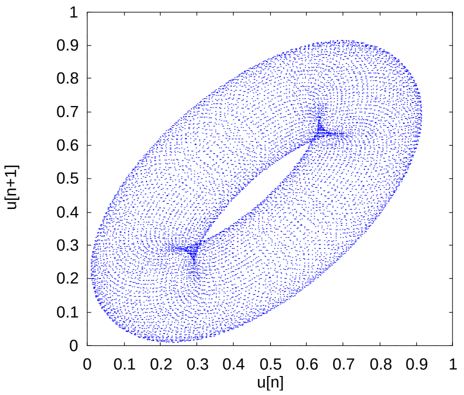

Figure 3. Trajectory of (2) with sinusoidal input of frequency 0.78 and amplitude 0.2. $\theta = 1.5$.

Once again, more complicated fractal trajectories can also be observed in the sinusoidally-driven system.

The effect of certain circuit non-idealities in the bandpass ΣΔ system is to place the zeroes of the noise transfer function at $\pm r.e^{j\theta}$ rather than at $\pm e^{j\theta}$.

$$u_n = 2r\cos\theta.u_{n-1} - r^2 u_{n-2} + 2r\cos\theta.(x_{n-1} - sgn(u_{n-1})) - r^2(x_{n-2} - sgn(u_{n-2})) \quad (3)$$

With zero input, as r is decreased below unity, the previously-elliptical trajectories become elliptical spirals converging to low-order periodic points. High-order periodic points become inadmissible as r is decreased, and the trajectories are increasingly likely to converge to low-order periodic points. Analysis of the effect of decreasing pole radius requires an understanding of the disappearance of these periodic points, and the changes in their basins of attraction. Figure 4 shows (in black) a portion of the basins of attraction of the period-two points of (3) with r = 0.996 and 2cosθ = 0.8 and zero input, in the $u_n\, u_{n+1}$ plane. The small ellipses visible within the structure are basins of attraction of different periodic points.

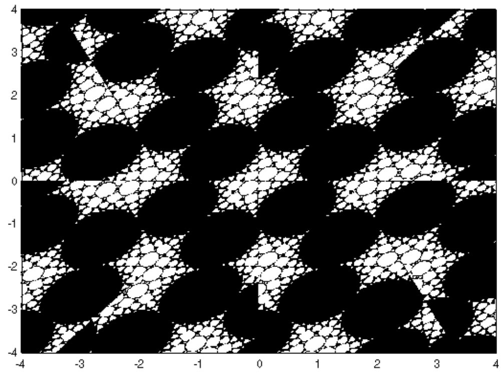

Figure 4. Basins of attraction in the $u_n u_{n+1}$ plane of the period-two points of the map given by (3) with zero input. r = 0.996 and $2\cos\theta = 0.8$.

A technique for predicting the shapes of these basins has developed, based on the theory of critical curves [13]. Use of this method, which is based on tracking the images and preimages of the lines of discontinuity of the system, results in figures such as that in Figure 5, which is the basin of attraction for one of the period-one points. The correspondence between the basins predicted by the theory in Figure 5, and those found by simulation in Figure 4 is clear. Similar analysis yields the basin of attraction of the period-two point, or indeed of any periodic point.

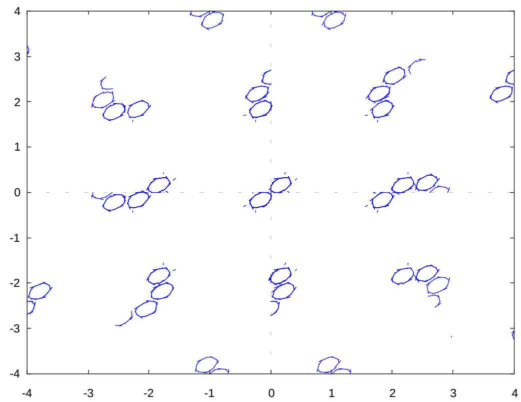

Figure 5. Basins of attraction in the $u_n u_{n+1}$ plane of one of the period-one points of the map given by (3) with zero input, as predicted using the critical curves tool. r = 0.996 and $2\cos\theta = 0.8$.

This method provides an explanation of the basins of attraction of the periodic points of the zero-input bandpass ΣΔ system. By studying the bifurcations of the basins as *r* decreases, the effect of certain component nonidealities on the behaviour of the system can be explained. Extension of this work to handle more general nonidealities, and to sinusoidal input, is in progress.

3. SUMMARY

ΣΔ modulation is an inherently nonlinear process, due to the presence of a nonlinear element (a one-bit quantiser) in the feedback loop. Conventional linear circuit analysis methods cannot provide a thorough understanding of this inherently nonlinear technique, leading a number of researchers to apply nonlinear system theory to the study of ΣΔ modulation. The aim of this paper has been to provide a summary of some of the results achieved by this work in the area of bandpass ΣΔ modulation. In particular, the effect on the behaviour of the system of decreasing pole radius, caused by component nonideality, can be ascertained by the use of the critical curves tool.

4. ACKNOWLEDGEMENTS

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