ILL-CONDITIONING OF NON-MINIMUM PHASE SYSTEMS

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ABSTRACT

The typical inverse problem is the recovery of the input, *x*, given data, *y* and the knowledge of the system *A*. Such problems occur frequently in instrumental science. For the Linear Time Invariant (*LTI*) systems the governing equation can be expressed in matrix form, *y=Ax*.

In this paper the problem of ill-conditioning of non-minimum phase systems and the relation of the phase structure of the system to the singular values of its system matrix is discussed.

1 INTRODUCTION

Considering the cases where the system is linear and time invariant, the relation between the measured data, *y*, and the input *x* is the convolution integral

$$y(t) = \int h(t-\tau)x(\tau)d\tau$$

where $h(t)$ is the transmission function of the forward system and we assume the kernel *h(t)* is known.

The outcome of any given experiment for determination of the input *x* specifies a set of measured values stored in a computer and the interpretation of the data requires the solution of a mathematical problem. Ignoring the effects of noise and the errors on the data, a preliminary formulation of a linear inverse problem is the construction of a continuous function *x* from the knowledge of a finite set of values, $y(t_i)$.

Generally the information about *x* is incomplete. One problem here is the lack of uniqueness of the solution, but when the number of data are large we might also face a lack of numerical stability.

2 CONDITIONING

A problem is said to be well posed if the three following requirements are fulfilled: *existence*, *uniqueness* and *stability* or continuity of the solution in the data. The latter case of stability is the one of primary interest in the study of inverse problems.

So an understanding of the effects of small errors in the representation of matrix problems and their numerical solution is vital. For example in the solution of *Ax=y*, where *A* and *y* are known and *A* is non-singular, the computer's approximation for the original *A* and *y* will be $\hat{A}$ and $\hat{y}$, and the actual problem being solved is

$$\hat{A}\cdot\hat{x} = \hat{y}$$

and the exact solution of this equation is different from *x*. Suppose that $\hat{y} = y$ then it can be shown [3] that the relative error in $\hat{x}$ is

$$\frac{\|\hat{x}-x\|}{\|x\|} \le \|A\|\cdot\|A^{-1}\|\cdot\frac{\|\hat{A}-A\|}{\|A\|} = K\cdot\frac{\|\hat{A}-A\|}{\|A\|}$$

where *K* is the condition number of the matrix *A*, *i.e.* a measure of the conditioning of the matrix.

The corresponding results for error in *y* is

$$\frac{\|\hat{x}-x\|}{\|x\|} \le K\cdot\frac{\|\hat{y}-y\|}{\|y\|}$$

So if *K* is large there is no way to guarantee that $\hat{x}$ will bear any relation to the true *x*. This phenomenon is called ***ill-conditioning*** and it makes the solution very sensitive to small changes in the data, and critically depends on the values of A_{ij}.

It is perhaps tempting to think that the invertibility of a matrix (i.e. its nearness to singularity) indicates a large condition number. This is not necessary so, for example, there is little correlation between the determinant of a matrix and its condition number.

As remarked in [1] and as we shall see later a determinant may be unity but the matrix be highly ill-conditioned, and conversely the determinant may be very small and the matrix not ill-conditioned.

A problem is said to be *well-conditioned* if small changes in parameters produce small changes in the solution.

3 INVERSION AND PHASE STRUCTURE

A general form for a system function is :

$$H(z) = \frac{A(z)}{B(z)} = \frac{k\prod_{j=1}^{N}\left(1 - a_j z^{-1}\right)}{\prod_{i=1}^{M}\left(1 - b_i z^{-1}\right)}$$

The roots of *B(z)* are the poles of *H(z)* and the roots of *A(z)* are its zeros. Inverting the *z*-transform of a signal its poles becomes zeros and vice versa, *i.e.*

$$F(z) = B(z) / A(z)$$

The stability and causality of inverse system *F(z)* depends on the location of its poles and zeros.

A minimum- phase polynomial is one that has all of its roots inside the unit circle.

Minimum-phase systems are causal linear shift-invariant systems with rational system functions where both *A(z)* and *B(z)* are minimum-phase polynomials. Since *H(z)* is stable if *B(z)* has all its roots inside the unit circle, thus a minimum-phase system is a causal, stable system with a causal stable inverse [2].

Maximum-phase systems has all their zeros of *H(z)* outside the unit-circle and a *non-minimum-phase* (*mixed-phase)* systems have some zeros inside and some outside the unit circle.

A finite length maximum-phase system has a system function [2],

$$H_{\max}(z) = z^{-(N-1)} H_{\min}(z^{-1})$$

where $H_{\min}(z)$ represent its equivalent minimum-phase system. This implies that

$$h_{\max}(n) = h_{\min}(N-1-n)$$

Let us consider the structure of system functions that are of mixed phase, an equivalent minimum-phase system can be formed [having identical modulus to *H(z)*] in the following way.

Any general system function may be written as $H(z) = H_{eq}(z) \cdot H_{ap}(z)$

Where $H_{ap}(z)$ is the system function of an all-pass filter and $H_{eq}(z)$ is the system function of equivalent system.

If we factorise *H(z)* into a *minimum* and *maximum* phase parts, it permits complete deconvolution,

$$H(z) = H_{\min}(z) \cdot H_{\max}(z)$$

where $H_{min}(z)$ and $H_{max}(z)$ represent the minimum and maximum phase part of *H(z)*. In the time domain

$$h(n) = h_{\min}(n) * h_{\max}(n)$$

$h_{min}(n)$ has a stable purely causal inverse, and $h_{max}(n)$ has a stable purely non-causal inverse. Each of these components are separately inverted, $h_{min}(n)$ has a stable causal inverse while $h_{max}(n)$ has a stable non-causal inverse

$$h^{-1}(n) = h_{\min}^{-1}(n) * h_{\max}^{-1}(n)$$

4 SINGULAR VALUE OF SYSTEM MATRIX

Assuming operation in the discrete time domain of a linear shift invariant system with input *x*, output *y* and impulse response *h*, the input output relation is,

$$y(n) = h(n) * x(n)$$

where ' $*$ ' denotes convolution, *i.e.*

$$y(n) = \sum_{m=-\infty}^{\infty} x(m) h(n-m)$$

For an *FIR*, causal *LTI* system we have

$$y(n) = \sum_{m=0}^{N} h(m) x(n-m) \qquad n \geq 0$$

where *y(0), y(1), y(2),......., y(N)* are the output data, *i.e.* convolution is a linear transformation and may be expressed in matrix form $\boldsymbol{y} = A\boldsymbol{x}$, where *A* is the system matrix. For *LTI* operator the system matrix has a Toeplitz structure and is lower triangular due to causality.

To analysis the system matrix we use one of the very powerful and useful matrix factorisation the “singular value decomposition” or *SVD* of a matrix. Every $(m \times n)$ matrix *A* can be expressed as a product of three matrices:

$$A = U\ S\ V^T$$

where *U* $(m \times m)$ and *V* $(n \times n)$ are both orthogonal matrices and *S* is a diagonal matrix whose diagonal element are called *singular values* of *A* [1].

The condition number of a matrix can be defined in term of singular values as $K = {s_{\max}}/{s_{\min}}$, so if a system matrix has relatively very small or very large singular values compare with the rest of its singular value, then the matrix is ill-conditioned and the reconstructed signal $\hat{x}$ can be far away from the true signal *x*.

Now consider any *N* point *FIR* filter *h(n)*, which may be written as

$$h(n) = h_1(n) * h_2(n) * h_3(n) * \cdots * h_{N-1}(n)$$

where each $h_i(n)$ is a two-point filter of the form $(1, -a_i)$. The system function of this two-point filter is, $H_i(z) = 1 - a_i z^{-1}$ which has a zero at $z = a_i$. We proceed by primarily considering just these simple two point filters [4]. The system functions of the forward and inverse systems are :

$$H(z) = 1 - a\ z^{-1} \text{ and } F(z) = \frac{1}{1 - a\ z^{-1}}$$

If $|a|<1$ the system is minimum phase and stable and has a causal, stable inverse, and if $|a|>1$ the system is maximum phase and its inverse is causal and unstable or is non-causal and stable.

For this 2-point *FIR* system with one zero at $z=a$, and for the case $m=n$, the forward system matrix and the inverse system matrix are

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots \\ -a & 1 & 0 & \dots \\ 0 & -a & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ a & 1 & 0 & \dots \\ a^2 & a & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We shall study the distribution of singular values of the system matrix using numerical simulations. It will demonstrate that this gives information about the "minimum phase or non-minimum phase" properties of the system. Note that the

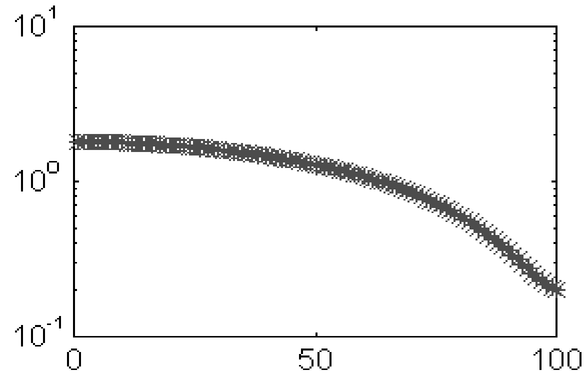

Figure (1): Singular values of minimum phase system

Figures 1 refers to a minimum phase system, note how the singular values are evenly distributed, leading to a small condition number.

Figures 2 refers to a maximum phase system, in this case *99* of the singular values are close to unity with a solitary one close to 10^{-10}.

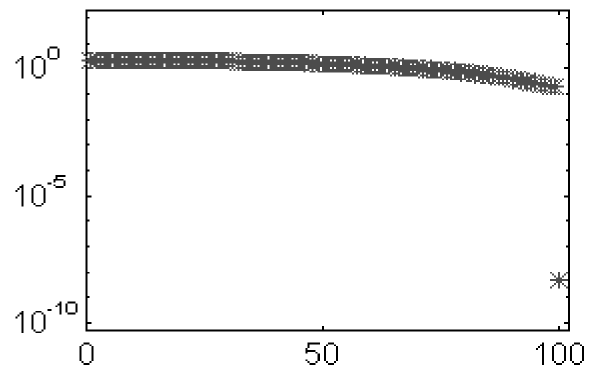

Figure (2): Singular values of maximum phase system

From Figure 2 we see that a maximum phase two-point filter gives rise to a system matrix with one singular value significantly smaller than the remainder. The large condition number for such a matrix can be attributed to this one singular value (smallest singular value).

This singular value gets smaller as the size of the matrix increases and as the zero moves away from the unit disc. In Figures 3 we trace how the smallest singular value varies with the value of *a*, it shows the change of smallest singular value of a two point filter system matrix with respect to the location of zero in the unit circle. As *a* exceeds unity the smallest singular value plummets.

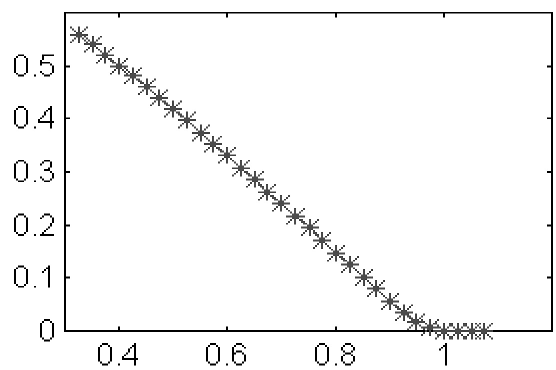

Figure (3): Change of smallest singular value with respect to change of zero location

The results of several simulation shows that if one considers more complex cases, with two or more zeros outside the unit circle the number of very small singular values are identical to the number of zeros outside the unit circle.

Figures 4 shows the singular values associated with system matrix of a maximum phase system which has two zeros outside the unit disc.

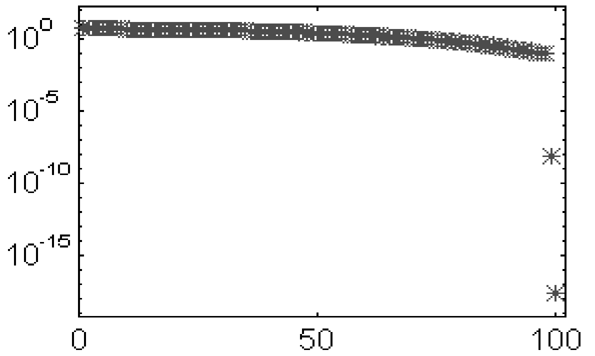

Figure (4): Singular values of a system with two zeros outside the unit circle

Let us now consider a simple *IIR* system which has a pole at *z=a, i.e.* $H(z)=\dfrac{1}{1-az^{-1}}$. If $|a|<1$ the system is stable and for $|a|>1$ the system is unstable.

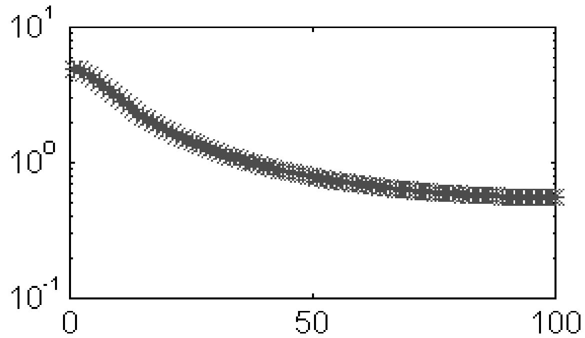

Figure (5): Singular values of truncated IIR system

In practice to study such a system using matrix methods its impulse response must be truncated and therefore an equivalent *N*-point *FIR* system with $N-1$ zeros is considered. Figure 5 considers the distribution of the singular values for such a two-point *IIR* filter.

If we move the pole location outside the unit circle and make the system unstable, Figure 6 shows that now it is the ***largest singular value*** that is much greater than the rest.

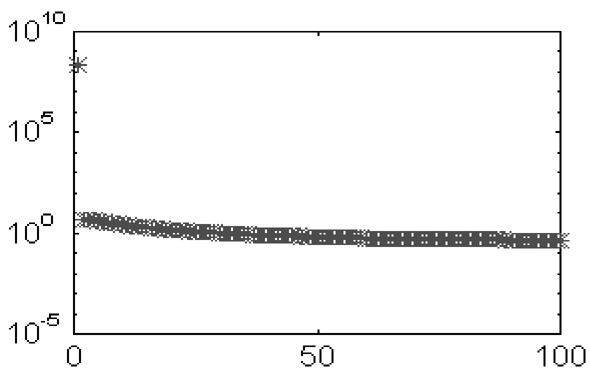

Figure (6): Singular values of the unstable system

5 Approaches to the Inversion of Non-Minimum Phase Systems

The small singular values, which relate to zeros outside the unit disc, lead to a system matrix with a large condition number, *K(A)*.

Regularisation is one approach to overcoming this problem, but we discuss two other methods here:

- The first uses a circulant matrix which is an approximation to the Toeplitz system matrix.
- The second is small singular value substitution.

5.1 Circulant Approximation:

One can construct circulant matrix which is "asymptotically equivalent" to *A* [5]. This circulant matrix is well conditioned, having almost the same singular values as *A*. The difference being that the circulant matrix does not have the very small singular values.

One then chooses to invert the well conditioned equivalent circulant matrix rather than *A*.

The eigenvectors of a circulant matrix are in fact the basis functions of an FFT and this approach effectively employs a FFT based solution.

5.2 Small Singular Value Substitution:

This is an alternative method for the inversion of an ill-conditioned system matrix [5]. The very small singular values, which cause the ill-conditioning, are replaced by an value representative of the clustered singular values, as follows:

- Calculate the singular values of *A*, using $A=USV^T$.
- Separate the very small singular values.
- Construct a matrix $\hat{S}$, substituting the smallest singular values with a value slightly smaller than the smallest remaining singular value.
- Invert the matrix $\hat{A}\,(=U\hat{S}V^T)$, which should now be well conditioned.

6 References

1. Golub G. and Van Loan C., 1989, *Matrix Computation*, John Hopkins University Press.
2. Oppenheim A.V. & R. W. Schafer 1975, *Digital Signal Processing*, Prentice-Hall.
3. Parker R.L. 1994, *Geophysical Inverse Theory*, Princeton.
4. Hashemi S. & J. K. Hammond 1994, Proceedings Of The Conference On Identification In Engineering Systems, pp. 194-204. *Recognition and Inversion of Non-minimum Phase System Using L_1 and L_2 Norms.*
5. Hashemi S. & J. K. Hammond 1996, Mechanical Systems And Signal Processing, Vol. 10, No. 3, *The Interpretation Of Singular Values In The Inversion Of Minimum And Non-Minimum Phase Systems.*