CHEBYSHEV DESIGN OF FIR FILTERS WITH ARBITRARY MAGNITUDE AND PHASE RESPONSES

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ABSTRACT

This paper presents a method for the design of nonlinear phase FIR digital filters with complex or real-valued coefficients using the Chebyshev error criterion. Three different problems are considered: Complex Chebyshev approximation with additional weighting of the resulting magnitude and phase errors, simultaneous Chebyshev approximation of a given magnitude and phase response, and simultaneous Chebyshev approximation of a given magnitude and group delay response. A linearization approach leads to a problem formulation that allows the use of stable algorithms with guaranteed convergence. It is shown that for this linear approach the simultaneous Chebyshev approximation of a desired magnitude and phase response is a special case of complex Chebyshev approximation with independent weighting of the magnitude and phase errors. Two existing design methods are included in this method as special cases.

1 INTRODUCTION

Several practical applications including equalization [1] and frequency-selective systems with arbitrary group delay [2] require the design of FIR filters with specified magnitude and phase responses. In many cases the minimization of the maximum error is an adequate optimization criterion. The resulting problem is to find the filter coefficients $h(n)$ $(n = 0, 1, \ldots, N-1)$ such that the Chebyshev norm of the considered error function is minimized. In practice, one or more of the following error functions are relevant:
The weighted complex error function

$$E_C(\theta) = W_C(\theta)[D(e^{j\theta}) - H(e^{j\theta})], \tag{1}$$

the weighted magnitude error function

$$E_M(\theta) = W_M(\theta)[|D(e^{j\theta})| - |H(e^{j\theta})|], \tag{2}$$

the weighted phase error function

$$E_\phi(\theta) = W_\phi(\theta)[\arg\{D(e^{j\theta})\} - \arg\{H(e^{j\theta})\}] \tag{3}$$

and the weighted group delay error function

$$E_\tau(\theta) = W_\tau(\theta)[\tau_d(\theta) - \tau(\theta)], \tag{4}$$

where $H(e^{j\theta}) = \sum_{n=0}^{N-1} h(n)e^{-jn\theta}$, $D(e^{j\theta})$ is the complex desired frequency response, $\tau_d(\theta)$ is the desired group delay response and $\tau(\theta)$ is the actual group delay response of the filter. The weighting functions are positive and real. All functions are defined on the domain $B \subseteq [0, 2\pi)$, which consists of one or more disjoint frequency intervals. For the design of filters with real-valued coefficients it is sufficient to consider the domain $B \cap [0, \pi]$.

The Chebyshev approximation problem for the design of filters with a specified complex frequency response can be stated in three different ways, depending on the error functions to be considered. In recent years much attention has been given to the weighted complex Chebyshev approximation. The corresponding problem is

$$\text{Minimize} \quad \max_{\theta \in B} |E_C(\theta)|. \tag{5}$$

Minimizing the Chebyshev norm of the complex error function results in a fixed interrelationship of the resulting magnitude and phase errors. For many practical applications it is desirable to specify this ratio in advance. Complex Chebyshev approximation does not minimize the magnitude and phase errors. If a direct minimization of these errors is desired, the problem must be stated as follows:

$$\text{Minimize} \quad \max_{\theta \in B} \{|E_M(\theta)|, |E_\phi(\theta)|\}. \tag{6}$$

The choice of the weighting functions $W_M(\theta)$ and $W_\phi(\theta)$ in (2) and (3) specifies the ratio of the resulting magnitude and phase errors. For some cases it is more practical to consider the group delay instead of the phase response of the filter to be designed. This approach leads to the following problem statement:

$$\text{Minimize} \quad \max_{\theta \in B} \{|E_M(\theta)|, |E_\tau(\theta)|\}. \tag{7}$$

In this paper, it is shown how the solutions of all three problems (5)–(7) can be approximated by linear methods. This allows the use of readily available and reliable standard software for the solution of the filter design problem. Additionally, it is shown that for this kind of linearization approach, simultaneous minimization of the magnitude and phase errors is a special case of complex Chebyshev approximation with weighting of the magnitude and phase errors.

2 THE DESIGN METHOD

Our goal is to derive a stable and simple to implement design procedure based on linear methods. All three problems (5)–(7) involve error functions that are nonlinear in the unknown filter coefficients. However, it is possible to find very good linear approximations for these error functions. The resulting linearization errors are negligible for most practical applications.

The magnitude of the complex error function can be expressed as follows:

$$|E_C(\theta)| = \max_{\alpha\in[0,2\pi)} \{\Re e(E_C(\theta)e^{j\alpha})\}. \quad (8)$$

This representation is linear in the unknown filter coefficients although it is still exact. However, an infinite set of angles $\alpha \in [0, 2\pi)$ has to be considered. Obviously, the representation

$$|E_C(\theta)| = \max_{\alpha\in[0,\pi)} \{|\Re e(E_C(\theta)e^{j\alpha})|\} \quad (9)$$

is equivalent to (8). Replacing the continous domain of α by a discrete set of equidistant angles

$$\alpha_i = \pi(i-1)/p, \quad i = 1, 2, \dots, p \quad (10)$$

leads to the definiton of

$$|E_C(\theta)|_p = \max_{1\le i\le p} \{|\Re e(E_C(\theta)e^{j\alpha_i})|\} \quad (11)$$

for any positive integer $p \geq 2$. It can be shown [3] that

$$|E_C(\theta)|_p \le |E_C(\theta)| \le |E_C(\theta)|_p/\cos(\pi/2p), \quad p \ge 2. \quad (12)$$

Hence, the approximation error δ_p that results from minimizing $\max_{\theta\in B} |E_C(\theta)|_p$ instead of minimizing $\max_{\theta\in B} |E_C(\theta)|$ can be bounded by

$$\delta_p \le \delta/\cos(\pi/2p), \quad (13)$$

where δ is the approximation error of the optimal solution of the complex Chebyshev approximation problem. The convergence rate of $\delta_p \to \delta$ is quadratic with $p \to \infty$.

2.1 Stopbands

In the stopbands $B_s = \{\theta \in B : |D(e^{j\theta})| = 0\}$ the problems (5)–(7) are equivalent, because there is no desired phase or group delay response in the stopbands and $|E_C(\theta)| = |E_M(\theta)|$ (for $W_C(\theta) = W_M(\theta) = W(\theta)$ which is no restriction). Hence in the stopbands the only goal is to minimize $|E_C(\theta)| = |E_M(\theta)| = W(\theta)|H(e^{j\theta})|$, where $W(\theta)$ is an arbitrary positive weighting function. Making use of the linearization (11) leads to the following problem:

$$\text{Minimize} \quad \max_{\theta\in B_s}\{W(\theta)|H(e^{j\theta})|_p\} = \max_{\theta\in B_s}\max_{1\le i\le p_s} \{W(\theta)|\Re e(H(e^{j\theta})e^{j\alpha_i^s})|\}, \quad (14)$$

where α_i^s are the angles defined in (10) for the desired linearization parameter $p = p_s$ in the stopbands. Choosing m_s discrete frequency samples $\theta_k^s \in B_s$ $(k = 1, 2, \dots, m_s)$ in the stopbands yields the following system of linear equations

$$W(\theta_k^s)\Re e\left\{H(e^{j\theta_k^s})e^{j\alpha_i^s}\right\} = 0, \quad (15)$$

$$i = 1, 2, \dots, p_s; \quad k = 1, 2, \dots, m_s.$$

This is the first part of an overdetermined system of linear equations which must be solved in the Chebyshev sense in order to obtain the desired filter coefficients.

2.2 Passbands

In the passbands $B_p = \{\theta \in B : |D(e^{j\theta})| > 0\}$ it is necessary to differentiate between the problems (5), (6) and (7). At first, a new linearized problem formulation for complex Chebyshev approximation with independent weighting of the magnitude and phase errors will be introduced and the problem (6) will be shown to be a special case of this formulation. Finally, the simultaneous minimization of the magnitude and group delay errors using linear methods will be discussed.

Simple geometrical considerations lead to the following linearized expressions for the magnitude and phase errors in the passbands:

$$E_M(\theta)/W_M(\theta) \approx \Re e\{H(e^{j\theta})e^{-j\phi_d(\theta)}\} - |D(e^{j\theta})|, \quad (16)$$

$$E_\phi(\theta)/W_\phi(\theta) \approx \Im m\{H(e^{j\theta})e^{-j\phi_d(\theta)}\}/|D(e^{j\theta})|, \quad (17)$$

$$\theta \in B_p.$$

Using these linearized expressions instead of the exact formulas leads to a solution of the approximation problem that is within second-order terms of the optimal solution, i. e. the results are "optimal to first order" [4]. A linear expression for the group delay error can easily be obtained from (17) by taking its negative derivative with respect to the frequency variable θ:

$$E_\tau(\theta)/W_\tau(\theta) \approx -\frac{d}{d\theta}\left(\Im m\{H(e^{j\theta})e^{-j\phi_d(\theta)}\}/|D(e^{j\theta})|\right),$$

$$\theta \in B_p. \quad (18)$$

Differentiating (17) yields a general linear expression which is valid for complex filter coefficients and arbitrary desired magnitude responses $|D(e^{j\theta})|$ in the passbands. For saving space, this rather lengthy formula is not stated here explicitly. A simplified version which is only valid for real-valued filter coefficients and a constant desired magnitude $|D(e^{j\theta})| = 1$ in the passbands was given in [2].

With these preliminaries the new linear approach for complex Chebyshev approximation with independent weighting of the magnitude and phase errors can be formulated. For the sake of simplicity, the weighting functions of the magnitude and phase errors be defined as $W_M(\theta) = w_M W(\theta)$ and $W_\phi(\theta) = w_\phi W(\theta)$ with constant w_M and w_ϕ. The generalization to a different frequency dependence of $W_M(\theta)$ and $W_\phi(\theta)$ is straight-forward. Define weights

$$w_i = \sqrt{w_M^2\cos^2\alpha_i^p + w_\phi^2\sin^2\alpha_i^p} \quad (19)$$

and angles

$$\tilde{\alpha}_i^p = \arctan\left(\frac{w_\phi}{w_M}\tan\alpha_i^p\right), \quad i=1,2,\ldots,p_p, \tag{20}$$

where α_i^p are the angles defined by (10) for the desired linearization parameter $p = p_p$ in the passbands. Choosing m_p discrete frequency samples $\theta_k^p \in B_p$ $(k = 1,2,\ldots,m_p)$ in the passbands and using the linearization formula (11) with an additional frequency dependent rotation according to the desired phase response $\phi_d(\theta) = \arg\{D(e^{j\theta})\}$ yields

$$w_i W(\theta_k^p)\Re e\left\{\left[D(e^{j\theta_k^p}) - H(e^{j\theta_k^p})\right]e^{-j(\tilde{\alpha}_i^p+\phi_d(\theta_k^p))}\right\} = 0,$$

$$i = 1,2,\ldots,p_p; \quad k=1,2,\ldots,m_p \tag{21}$$

from which

$$w_i W(\theta_k^p)\Re e\left\{H(e^{j\theta_k^p})e^{-j(\tilde{\alpha}_i^p+\phi_d(\theta_k^p))}\right\} = w_i W(\theta_k^p)|D(e^{j\theta_k^p})|\cos\tilde{\alpha}_i^p,$$

$$i = 1,2,\ldots,p_p; \quad k=1,2,\ldots,m_p \tag{22}$$

follows.

Let $p_p \geq 2$ be an even integer. For fixed indices $i = 1$ and $i = p_p/2+1$ the equations (22) read

$$w_M W(\theta_k^p)\Re e\left\{H(e^{j\theta_k^p})e^{-j\phi_d(\theta_k^p)}\right\} = w_M W(\theta_k^p)|D(e^{j\theta_k^p})|$$

$$k = 1,2,\ldots,m_p \tag{23}$$

and

$$w_\phi W(\theta_k^p)\Im m\left\{H(e^{j\theta_k^p})e^{-j\phi_d(\theta_k^p)}\right\} = 0,$$

$$k = 1,2,\ldots,m_p, \tag{24}$$

respectively. Comparing these equations with the linearized expressions (16) and (17) for the magnitude and phase errors shows that (23) and (24) result from weighting and evaluating (16) and (17) at the frequency points θ_k^p $(k = 1,2,\ldots,m_p)$ and setting them equal to zero. Additionally, the phase error is weighted with $|D(e^{j\theta})|$ which implies that for small $|D(e^{j\theta})|$ the optimization of the magnitude error becomes dominant.

Combining the systems of linear equations (15) and (22) gives the total overdetermined linear system which must be solved in the Chebyshev sense in order to obtain the desired filter coefficients. This is a standard linear programming problem. However, there are several robust algorithms which are specially suited to this problem (e. g. [5, 6]). In addition to complex Chebyshev approximation, arbitrary weighting of the resulting magnitude and phase errors can be achieved by appropriate choice of the weights w_M and w_ϕ. As already shown, the equations for minimizing the magnitude and phase errors are automatically included into the system. Hence, simultaneous approximation of a given magnitude and phase response results as a special case for the choice $p_p = 2$. In this case the system (22) is equivalent to the equations (23) and (24).

For simultaneous Chebyshev approximation of a desired magnitude and group delay response, a system of linear equations for the passbands is obtained in an analogous manner by weighting and evaluating the linearized expressions (16) and (18) at m_p frequency points θ_k^p $(k = 1,2,\ldots,m_p)$ and setting them equal to zero.

3 RELATION TO PREVIOUS WORK

There have been mainly two linearization approaches for the design of nonlinear phase FIR filters in the Chebyshev sense [2, 4]. Both reduce the design problem to the problem of solving an overdetermined system of linear equations in the Chebyshev sense or, equivalently, to a linear programming problem. For the case of ordinary complex Chebyshev approximation without weighting of the magnitude and phase errors $(w_M = w_\phi)$ the method proposed in this paper is equivalent to the approach presented in [2] which goes back to a method by Streit and Nuttall [3]. In [2] the authors proposed an additional weighting of the phase error by adding linear equations to the overdetermined system. In a similar manner an additional weighting of the group delay error was proposed. Although it was not proposed in [2], it would be possible to add linear constraints for an additional weighting of the magnitude error so that a true relative weighting of the magnitude and phase errors could be realized. The disadvantage of this approach is the large size of the resulting overdetermined system. The method proposed in this paper allows complex Chebyshev approximation with additional weighting of the phase error as well as the magnitude error without increasing the size of the overdetermined system. Concerning the case of additional optimization of the group delay response, it was observed that simultaneous approximation of the desired magnitude and group delay responses yields results similar to those obtained by the method proposed in [2] with the advantage of resulting in a smaller overdetermined system.

In [4] the Chebyshev design of FIR allpass filters is considered. For this purpose the linear expressions (16) and (17) for the magnitude and phase errors, specialized for real filter coefficients and $|D(e^{j\theta})| = 1$, are used to formulate a linear programming problem. It can easily be shown that this problem formulation is equivalent to the method given in this paper for the case $p_p = 2$ and $|D(e^{j\theta})| = 1$, given that there are no stopbands and that the desired phase response satisfies $\phi_d(\theta) = -\phi_d(-\theta)$.

In [1] an interpolative procedure for complex Chebyshev approximation, analogous to the Parks-McClellan algorithm for the linear phase case was presented. Additionally, the author proposed a method to allow relative weighting of the magnitude and phase errors. This is achieved by a transformation of the complex error function. It can be shown that the linearization approach for complex Chebyshev approximation presented by Streit and Nuttall [3] in conjunction with this transformation of the error function results in a problem formulation which is closely related to the one given in this paper. The important advantage of using this transformation in conjunction with the linearization method instead of any other method is the possibility to arbitrarily choose the linearization parameter p_p as well as the weigths w_M and w_ϕ. This allows an arbitrary trade-off between complex Chebyshev approximation with optional weighting of the magnitude and phase errors and simultaneous Chebyshev approximation of the desired magnitude and phase responses.

4 DESIGN EXAMPLES

Example 1: Lowpass Filter with Weighted Phase Error
Considering the design of a nearly linear phase lowpass filter of length $N = 50$ with real-valued coefficients, with a passband $\theta \in [0, 0.6\pi]$ and a stopband $\theta \in [0.67\pi, \pi]$, the design method presented in this paper is compared with the method proposed in [2]. The passband weighting is 1 and the stopband weighting is 10. The desired group delay in the passband is 10 samples. An additional weighting of the phase error with $w_\phi = 2$ is desired. $D(e^{j\theta})$ is evaluated at 300 frequency points distributed uniformly over the passband and the stopband. Table 1 shows the design results for the method given in [2] ($p = 6$) and for the new method ($p_s = 6$). The errors $E_{M,p}$ and $E_{M,s}$ denote the magnitude errors in the passband and in the stopband, respectively. The dimension of the resulting problem (number of equations $\times$ number of unknowns) is denoted by dim. It turned out that choosing $p_p = 2$ yields a solution that is even slightly superior to the one obtained by the method presented in [2] with $p = 6$. This is due to the fact that if equations for an additional weighting of the phase error are added to the overdetermined system, many of the other equations become redundant and could be eliminated. Hence, equivalent results are obtainable with a considerably smaller system of equations. Due to the smaller dimension of the problem, the memory requirement as well as the computational effort for computing the desired filter coefficients are reduced considerably.

Example 2: Bandpass Filter with Weighted Group Delay Error
This design example was originally introduced in [2]. It is a bandpass filter of length $N = 31$, with two stopbands $\theta \in [0, 0.2\pi]$ and $\theta \in [0.66\pi, \pi]$, and a passband $\theta \in [0.3\pi, 0.56\pi]$. The passband weighting is 1 and the stopband weightings are 10. The desired group delay is 12 samples. An additional weighting of the group delay error with $w_\tau = 1$ is desired. $D(e^{j\theta})$ is evaluated at 200 frequency points distributed uniformly over the passband and the stopbands. Table 1 shows the design results for both design methods with $p = 8$ and $p_s = 8$, respectively. For solving the design problem with the new method, simultaneous minimization of the group delay and magnitude errors was chosen. As shown in table 1, the design results are equivalent to those obtained by complex Chebyshev approximation with additional weighting of the group delay error. The latter approach results in an unnecessarily large overdetermined system.

	Chen&Parks [2]		new method	
	ex. 1	ex. 2	ex. 1	ex. 2
$E_{M,p}$	8.88 e-2	1.12 e-1	8.74 e-2	1.11 e-1
$E_{M,s}$	9.01 e-3	1.12 e-2	8.69 e-3	1.12 e-2
E_ϕ	4.70 e-2	1.90 e-2	4.58 e-2	2.15 e-2
E_τ	1.40	0.17	1.18	0.16
dim	1993×50	1665×31	1028×50	1210×31

Table 1: Design results for example 1 (ex. 1) and example 2 (ex. 2) comparing the new method with the method given in [2].

5 CONCLUSION

The proposed linear design method unifies the solutions of the three different Chebyshev approximation problems occurring in practical nonlinear phase FIR filter design. Although all problems under consideration are nonlinear in the unknown filter coefficients, the proposed linear approach is shown to yield highly satisfactory results. Compared to other linear methods, the computational effort can be reduced considerably for most practical design problems when the new method is used.

References

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