DETERMINISTIC ESTIMATION OF THE BISPECTRUM AND ITS APPLICATION TO IMAGE RESTORATION

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ABSTRACT

While the bispectrum has desirable properties in itself and therefore has a lot of potential to be applied to image restoration, few real-world application results have appeared in the literature. The major problem with this is the difficulty in realizing the expectation operator, due to the lack of realizations. In this paper, the true bispectrum is defined as the expectation of the sample bispectrum, which is the Fourier representation of the triple correlation given one realization. The characteristics of sample bispectrum are analyzed and a way to obtain an estimate of the true bispectrum without stochastic expectation, using the generalized theory of weighted regularization is shown.

1 INTRODUCTION

There has been a considerable amount of work in the literature on the estimation of higher order spectra, especially the bispectrum [1]-[4]. However, a major problem with a number of real-world applications is the difficulty in realizing the expectation operator since only a very limited number of realizations of the signal field is available. Furthermore, even if the expectation operator is approximately performed based on the signal assumption of local or quasi-ergodicity, the computation is too expensive to be used for practical applications, since the dimension is doubled when the signal is transformed into the bispectral or third order cumulant domain.

In this paper, the sample bispectrum is defined as the Fourier domain representation of the triple correlation given one realization of the ensemble. The true bispectrum is, therefore, defined as the expectation of the sample bispectrum [4]. After the characteristics of the sample bispectrum are analyzed, an efficient way to obtain an estimate of the true bispectrum from one realization or very limited number of realizations is shown based on the generalized theory of weighted regularization [5].

2 ANALYSIS OF THE SAMPLE BISPECTRUM OF A NOISY DEGRADED SIGNAL

Consider the following signal degradation model (spatially invariant blur)

$$y(m) = d(m) * x(m) + n(m), \tag{1}$$

where $*$ denotes convolution and m is the discrete time or discrete spatial coordinates. $y(m)$, $x(m)$ and $n(m)$ represent the observed signal, the original signal and the additive noise, respectively, and $d(m)$ the impulse response of the degradation system. Here $x(m)$ and $d(m)$ are considered deterministic signals, but $n(m)$ is a stochastic signal, resulting in a stochastic $y(m)$.

Given the observed noisy degraded signal $y(m)$ of Eq. (1), we want to find an estimate of the true bispectrum of $x(m)$ based on the discrete bispectrum $B_y(k_1,k_2)$ of $y(m)$ with the prior knowledge of $B_d(k_1,k_2)$ of $d(m)$, which can be directly calculated from the DFT of $d(m)$ since it is deterministic. The true bispectrum of $y(m)$ is equal to

$$B_y(k_1,k_2) = E[B_{ys}(k_1,k_2)], \tag{2}$$

where B_{ys} is the sample bispectrum of one available realization of $y(m)$ ensemble, i.e., defined by $Y(k_1)Y(k_2)Y^*(k_1,k_2)$. The sample bispectrum of $y(m)$ can be rewritten, based on Eq. (1), as [4]

$$B_{ys}(k_1,k_2) = Z(k_1)Z(k_2)Z^*(k_1+k_2) + NZ(k_1,k_2) \tag{3}$$

where $NZ(k_1,k_2) = Z(k_1)Z(k_2)N^*(k_1+k_2) + Z(k_1)N(k_2)Z^*(k_1+k_2) + N(k_1)Z(k_2)Z^*(k_1+k_2) + Z(k_1)N(k_2)N^*(k_1+k_2) + N(k_1)Z(k_2)N^*(k_1+k_2) + N(k_1)N(k_2)Z^*(k_1+k_2) + N(k_1)N(k_2)N^*(k_1+k_2)$. Therefore,

$$B_y(k_1,k_2) = Z(k_1)Z(k_2)Z^*(k_1+k_2) + E[NZ(k_1,k_2)], \tag{4}$$

where $E[NZ(k_1,k_2)] = NX(0)[S_{nn}(k_1)\delta(k_1+k_2) + S_{nn}(k_2)\delta(k_1) + S_{nn}(k_1)\delta(k_2)]$, with $Z(k) = D(k)X(k)$ and $D(k)$, $X(k)$ and $N(k)$ the DFT sequences of $d(m)$, $x(m)$ and $n(m)$, respectively. S_{nn} denotes the power spectrum of the noise. It is shown in (4) that the expectation of B_{ys} results in the true bispectrum except the bispectral axes (if the mean value of $y(m)$ is not zero) or in the entire domain (if the mean is zero). However, since we have just one or a very limited number of realizations of $y(m)$ in many of the practical situations, the expectation of the sample bispectrum cannot be performed, but the sample bispectrum of the observed noisy signal is available. Thus, as shown in (3), the sample bispectrum consists of seven non-white noise terms besides the true bispectrum of the degraded signal without noise. Due to the limited number of realizations, the delta functions positioning the corrupted lines become sinc functions with the width of the main lobe being inversely proportional to the number of available realizations. Therefore, the neighboring region of the three axes ($k_1 = 0$, $k_2 = 0$, and $k_1 = -k_2$) are also corrupted. The farthest region which is least affected by noise is the diagonal line ($k_1 = k_2$).

If we consider this axis only ($k_1 = k_2 \equiv k$), then the sample bispectrum (3) can be written as,

$$B_{ys}(k) = B_{dd}(k)B_{xd}(k) + N(k)H(k), \qquad (5)$$

where $B_{dd}(k) = D(k)D(k)D^*(2k)$ and $B_{xd}(k) = X(k)X(k)X^*(2k)$, the true bispectrum of x to be estimated. $H(k)$ is defined by $Z(k)[Z(k) + 2Z^*(2k)]$, denoting the spectral shaping function which colorizes the white noise $n(m)$ in the bispectral domain. On this axis, we have just one independent variable k instead of k_1 and k_2. Here we assume that the second and higher order terms of the noise spectra $N(k)$ in the sample bispectrum are much smaller than the first order terms so there terms can be ignored. If we take the inverse DFT of $B_{ys}(k)$ in (5), then we obtain the following equation in the triple correlation domain

$$t_{ys} = T_{dd}t_{xd} + H(x)n, \qquad (6)$$

where the vectors t_{ys}, t_{xd} and n are the lexicographically ordered vectors of the inverse DFTs of B_{ys}, B_{xd}, and $N(k)$, respectively and T_{dd} and $H(x)$ are the block circulant matrices which compose the inverse DFTs of $T_{dd}(k)$ and $H(k)$, respectively. Since the first quadrant of the bispectral domain is symmetric with respect to the $k_2 = -k_1 + N/2$ line, only half of the required number of the bispectral frequency components are available on this axis. In other words, even frequency components of the bispectrum can be reconstructed from this diagonal axis.

The off-diagonal axis $k_1 = k_2 - 1 \equiv k$ which is closest to the diagonal axis is also least affected by the noise power terms concentrated on the three axes ($k_1 = 0$, $k_2 = 0$ and $k_1 = -k_2$). On this axis, the sample bispectrum and its triple correlation are represented, based on the same procedure mentioned above, by

$$B_{ys}(k) = B_{do}(k)B_{xo}(k) + H(k)N(k), \qquad (7)$$

where $B_{do}(k) = D(k)D(k+1)D^*(2k+1)$ and $B_{xo}(k) = X(k)X(K+1)X^*(2k+1)$ and

$$t_{ys} = T_{do}t_{xo} + H(x)n, \qquad (8)$$

where the vectors t_{ys} and t_{xo} are the lexicographically ordered vectors of the inverse DFTs of $B_{ys}(k)$ and $B_{xo}(k)$, respectively, and T_{do} and $H(x)$ are the block circulant matrices which compose the inverse DFTs of $T_{do}(k)$ and $H(k)$, respectively. The restoration of the true bispectrum on this axis provides the odd components of the bispectrum.

With the sample bispectrum equations or the sample triple correlation equations on the two axes, (6) and (8), all the components required for the full recovery of the true bispectrum are provided. The major advantage of this analysis is as follows : The bispectrum or the corresponding triple correlation intrinsically has the same dimension as that of the signal, unlike that the dimension of the conventional bispectrum (or triple correlation) which is doubled.

However, it is obvious that the matrices T_{xd} and T_{xo} are singular, and thus, the recovery of the true bispectrum becomes an ill-posed problem. Furthermore, the bispectral shaping function matrix $H(x)$ is signal dependent, which makes this inverse problem more nonlinear. Therefore, the generalized weighted regularization approach is adopted for this nonlinear inverse problem.

3 GENERALIZED WEIGHTED REGULARIZATION FOR RECOVERY OF TRUE BISPECTRUM

Let us use the notation t_x and T_d to denote t_{xd} and T_{dd}, or t_{xo} and T_{do}, for convenience. The additive noise n in the degradation model (1) is assumed to be white Gaussian.

In Eqs. (6) and (8), the recovery of bispectrum t_x from t_y is an ill-posed problem, since the T_d is singular in most cases. Furthermore, the spectral shaping function ($H(x)$) of the noise n in the sample triple correlation domain is a function of the original signal x, which is not available.

When the additive noise terms in (6) and (8) are signal-independent white Gaussian, minimization of an l_2 norm based smoothing functional may be used for obtaining a solution (t_x). This is justified by the relationship between the stochastic Maximum A Posteriori (MAP) approach and the deterministic l_2 norm based approach. However, since the additive noise term in the sample triple correlation domain is not signal-independent, we propose to obtain an estimate of the true bispectrum of x and thus true triple correlation (t_x) by minimizing the weighted smoothing functional given by

$$M_h(\alpha_h(t_x), t_x) = \|t_{ys} - T_d t_x\|^2_{A(t_x)} + \alpha_h(t_x)\|Ct_x\|^2, \qquad (9)$$

where $A(t_x)$ is a weighting matrix used to incorporate the noise characteristics in the sample triple correlation domain, $\alpha_h(t_x)$ is the regularization functional used to control the trade-off between fidelity to the data and smoothness of the solution, and C represents a high pass filter formed by a 2-D Laplacian kernel. The regularization parameter $\alpha_h(t_x)$ is defined as a functional of t_x, in order to make the weighted smoothing functional in (9) convex and have a unique global minimizer, as described in [5, 7]. Various choices of the functional $\alpha_h(t_x)$ can potentially result in meaningful minimizers of $M_h(\alpha_h(t_x), t_x)$. In our previous work [5], the three properties of $\alpha_h(t_x)$ and $M_h(\alpha_h(t_x), t_x)$ are considered necessary for such an objective. After investigating the desirable properties the regularization functional needs to satisfy, the following form has been shown to provide optimal solutions:

$$\alpha_h(t_x) = \frac{\|t_{ys} - T_d t_x\|^2_{A(t_x)}}{1/\gamma - \|Ct_x\|^2}, \tag{10}$$

where γ is the control parameter for convexity and existency of solution.

Using a stochastic Bayesian interpretation of the smoothing functional, the weighting matrix $A(t_x)$ can include the signal dependent non-white characteristics of the noise as described in [6] Therefore, the major challenge is how to incorporate this signal dependent spectral shaping function into our solution procedure without any prior knowledge about the original signal and thus $H(x)$.

Assuming $h(x)$ is locally ergodic, let us define first the covariance matrix of $h(x)$ as

$$\Theta_h(x) = R_{h(x)} + QR_{h(x)}Q^T - R_{h(x)}Q^T - QR_{h(x)}, \tag{11}$$

where $m_{h(x)}$ denotes the mean of $h(x)$ and $R_{h(x)} = Qh(x)h^T(x)$ is the autocorrelation matrix of $h(x)$. Motivated by a Bayesian interpretation of the weighted smoothing matrix, $A(t_x)$ is determined in terms of $\Theta_h(x)$ by $A(t_x) = I - \frac{\Theta_h(x)}{\gamma}$, where γ is chosen so that $\gamma \geq \max_i \theta_{h\,i}(x)$ and $\theta_{h\,i}(x)$ denotes the i-th element of matrix $\Theta_h(x)$.

With this choice of the weighing matrix and the regularization functional, the equation for the solution obtained by setting the gradient of $M_h(\alpha_h(t_x), t_x)$ in (9) equal to zero is given by

$$(T_d^T A(t_x)T_d + \alpha_h(t_x)C^T C)t_x = T_d^T A(t_x)t_{ys}. \tag{12}$$

since $\nabla_{t_x}\alpha_h(t_x) = 0$, as similarly described in [7].

4 ITERATIVE SIMULTANEOUS TRUE BISPECTRUM ESTIMATION AND SIGNAL RECOVERY

Since Eq. (12) is nonlinear and no prior information about the regularization functional and the weighting matrix is available, it is solved iteratively. The successive approximations iteration to solve (12) is written as

$$t_{x_{k+1}} = t_{x_k} + T_d^T A(t_{x_k})t_{ys} - (T_d^T A(t_{x_k})T_d + \alpha_h(t_{x_k})C^T C)t_{x_k}. \tag{13}$$

The sufficient condition for convergence of iteration (13) combined with the convexity of the weighted smoothing functional (9) is

$$\frac{1}{\gamma} \geq \|t_{ys}\|^2. \tag{14}$$

If condition (14) is satisfied, then the iterative algorithm (13) becomes independent of initial conditions in spite of its nonlinearity, producing the unique recovered true bispectrum of the original signal x along the two axes ($k_1 = k_2$ and $k_1 = k_2 - 1$).

At each iteration step of (13), the line triple correlation t_{x_k} which is represented in the frequency domain as the line bispectrum along the diagonal axis ($k_1 = k_2$) and the off-diagonal axis ($k_1 = k_2 - 1$) is obtained. Simultaneously with the partially recovered bispectrum of x (or triple correlation), the original signal at the k-th iteration step, x_k, can be obtained from the restored line bispectrum at that step, by recursion. The signal can be reconstructed from the estimated true bispectrum by estimating the phase and the magnitude components of the signal from the bispectral phase and the bispectral magnitude, respectively, and then combining the phase and the magnitude [4].

5 EXPERIMENTAL RESULTS AND DISCUSSIONS

A signal of 256 samples, which represents an arbitrary chosen line of an image (Fig. 1), is degraded by noise so that the resulting SNR is equal to 20dB (Fig. 2). The reconstructed signal lines from the bispectrum (triple correlation) estimated by the proposed algorithm and by the sample bispectrum without any noise removal, with one realization of the noisy signal line, are shown in Figs. 3 and 4, respectively. Apparently, the reconstruction of the signal line from the bispectrum requires that the expectation operator be implemented using a number of noisy realizations. Since just one realization is given, the signal reconstructed from the noisy signal line is not acceptable (Fig. 4). On the other hand, the reconstructed signal line by the proposed algorithm shows a very good result, even though just one realization is used. As was explained in the previous section, the deterministic weighted regularization successfully removes the colored noise term from the sample bispectrum in (3). Figure 5 shows the improvement in SNR of the reconstructed signal versus the number of the available realizations, by the weighted regularization algorithm and by performing the expectation operator. This plot shows that the reconstructed signals by the proposed algorithm are superior to those obtained by performing the expectation operator up to when about 50 realizations of the noisy signal is available. Since in many practical situations only a finite number of realizations are available (for example, 1 to 10 realizations), it is seen that the weighted regularization algorithm can replace the expectation operator for estimating the true

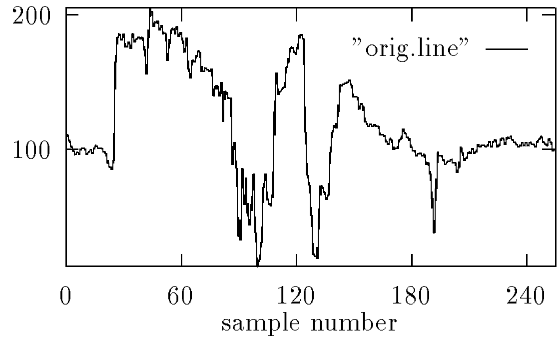

Figure 1: original line (256pts)

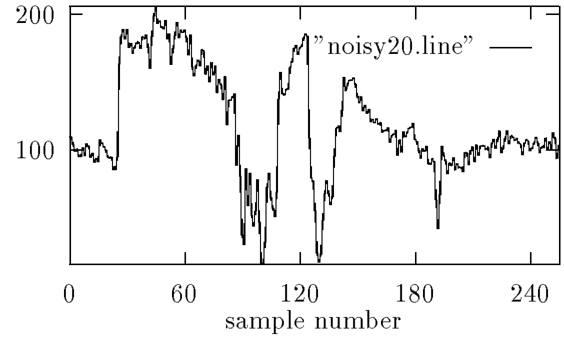

Figure 2: noisy line (20dB SNR)

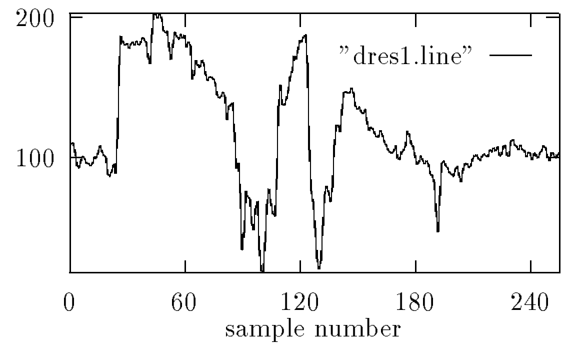

Figure 3: restored noisy line by new bispectral filter : 1 realization

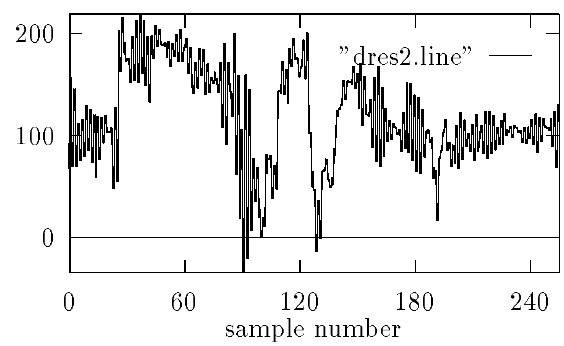

Figure 4: restored noisy line by conventional bispectral filter : 1 realization

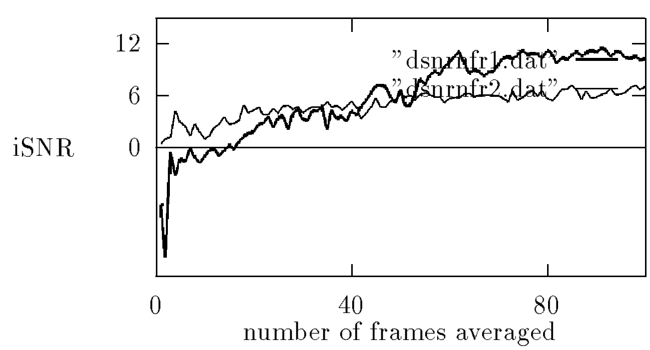

Figure 5: Δ_{SNR} .vs. # of frames averaged (20dB noisy case)

bispectrum. It is also evident from Fig. 5 that the result obtained by the proposed algorithm with one realization available are matched, in terms of SNR improvement, by the conventional algorithm when about 20 realizations are available.

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