A HIGHER-ORDER CUMULANT BASED DOA ESTIMATION ALGORITHM

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ABSTRACT

Most of the existing direction-of-arrival estimation algorithms depend on decomposition of the covariance matrix of the system which in turn require modeling of the contaminating noise. In this paper, a higher-order cumulant based algorithm for estimating the direction-of-arrival of m narrowband far field sources impinging on an array with n uniformly spaced sensors is proposed. Due to the unique property of higher order cumulant, the proposed method is shown to be at least theoretically independent of the additive Gaussian noise. The algorithm first evaluates the $2r^{th}$ order cumulant from the output of the system. By making use of these output cumulants, we obtain a new vector in which its elements are the coefficients of an equation whose roots are the DOA of the sources. The validity of the algorithm is demonstrated by extensive computer simulations.

1 INTRODUCTION

Estimation of channel matrix or identification of unknown sources from an observed signal array sensor have attracted much attention recently[1-6]. The objective of channel estimation is to derive the channel matrix from the measurement which is usually corrupted by additive noise. One of the applications of channel estimation is to estimate the Direction-Of-Arrival (DOA) of a target. Many different approaches have been proposed to tackle the DOA estimation problem in the past decade. A well-known class of methods, including the maximum likelihood method[1], MUSIC[2] and ESPRIT[3], are based on decomposing the covariance matrix of the sensor outputs that results in two orthogonal subspaces, namely the signal subspace and the noise subspace. The DOAs are then estimated by locating the minima of a certain cost function. Estimation of the unknown parameters by these approaches can be efficacious if at least one of the following conditions is satisfied. The additive noise is either spatially uncorrelated or white or colored with known covariance. Unfortunately, such information is not always available *a priori*. Recently, algorithms based on higher-order cumulant have been designed such as the MUSIC-like and minimum variance estimators proposed by Porat and Friedlander[4] and the one reported by Shamsunder and Giannakis[5]. Swami and Mendel[6] also considered cumulant-based algorithms for retrieval of harmonics under noisy environment as well as for DOA estimation problems.

The motivations behind using higher-order cumulants include: firstly, cumulants of Gaussian distributed processes higher than second order, irrespective to whether the processes are white or colored or correlated, are identically zero. Secondly, the cumulant of the sum of linearly superimposed signals is the sum of the cumulant of individual sources if the source signals are all independent to each other. Therefore, a noisy system can be treated as one operated in a noise free environment if the additive noise is Gaussian distributed. Hence, the estimation can be simplified since neither modeling of additive Gaussian noise nor estimation of the noise covariance matrix is needed. However, nonlinear optimization is necessary, which is rather complicated and computational intensive, to obtain the solution[4,5,11]. In this paper, we propose a cumulant based algorithm for DOA estimation that does not require complex nonlinear optimization but simple root finding that function properly even under very noisy environment.

2 THE ESTIMATION ALGORITHM

2.1 System Model

The noise free n-sensor-m-source system is given by

$$\mathbf{y}(k) \;=\; \mathbf{A}(\theta)\,\mathbf{s}(k) \tag{1}$$

where $\mathbf{s}(k) = [s_1(k)\ s_2(k)\ \cdots\ s_m(k)]$ is the source vector and $\mathbf{A}(\theta)$ is the parameter matrix containing the DOA information

$$\mathbf{A}(\theta) \quad = \quad [\mathbf{a}(\theta_1)\ \mathbf{a}(\theta_2)\ \cdots\ \mathbf{a}(\theta_m)] \tag{2}$$

where $\mathbf{a}(\theta_i) = [e^{-\jmath\phi_0(\theta_i)}\ e^{-\jmath\phi_1(\theta_i)}\ \cdots\ e^{-\jmath\phi_{n-1}(\theta_i)}]^T$ and $\phi_k(\theta_i) = c \cdot k \cdot \sin\theta_i$ for some constant c which is characterized by the sensor array. When noise is present, the system output, $\mathbf{x}(k)$, can be written as

$$\mathbf{x}(k) \;=\; \mathbf{A}(\theta)\,\mathbf{s}(k) \;+\; \mathbf{w}(k) \tag{3}$$

where $\mathbf{w}(k)$ is a complex Gaussian noise vector. The task is to estimate the parameter matrix, which is also known as the DOAs, of the system from the noise corrupted data $\mathbf{x}(k)$. Let us first introduce some basic assumptions in which the algorithm is based on:

A.1. $\mathbf{s}$ is a vector of $m \times 1$ narrowband, zero-mean and non-Gaussian sources with finite $2r^{th}$-order cumulant.

A.2. $\mathbf{w}$ is a vector of zero-mean additive Gaussian noise with unknown covariance matrix.

A.3. The signal sources are mutually independent.

A.4. $\mathbf{s}$ and $\mathbf{w}$ are statistically independent.

It is intriguing to note that the requirement of the covariance of $\mathbf{w}(k)$ to satisfy $\mathrm{E}\{\mathbf{w}\mathbf{w}^H\} = \sigma^2\mathbf{I}$ as in the subspace methods is not needed in the proposed algorithm.

2.2 Cumulants of Complex Processes

Although the k^{th} order cumulants of real processes are uniquely defined, this is not true for complex processes. There are 2^k possible representations of complex cumulants[7] which might affect the performance of the estimation algorithm. The problem of complex cumulant relies on the number of signals to be conjugated and which signal(s) should be conjugated. Swami and Mendel[6] defined the complex cumulant by conjugating exactly half of the total number of signals. The fourth cumulant of s_1, s_2, s_3 and s_4 is defined as $\mathrm{cum}\{s_1^*, s_2^*, s_3, s_4\}$. In our work, we adopt the above definition of cumulant for complex processes. Moreover, the odd-order cumulants of zero-mean symmetrically distributed signals are zero. Consequently, even order cumulants are used throughout the discussion.

2.3 The Proposed Algorithm

Denote the $2r^{th}$ cumulant of the sensor outputs $x_{l_1}, \cdots, x_{l_{2r}}$ by $C_{l_1 l_2 \cdots l_{2r}} = \mathrm{cum}\{x_{l_1}^* \cdots x_{l_r}^* \, x_{l_{r+1}} \cdots x_{l_{2r}}\}$ which is given by

$$C_{l_1 l_2 \cdots l_{2r}} = \sum_{i=1}^{m} \prod_{j=1}^{r} b_i^{l_j} \cdot c_i \tag{4}$$

where

$$b_i = \begin{cases} a_i^* & \text{if} \quad i \le r \\ a_i & \text{if} \quad i > r \end{cases}$$

and $c_i = \mathrm{cum}\{\overbrace{s_i^*, \cdots, s_i^*}^{r}, \overbrace{s_i, \cdots, s_i}^{r}\}$. It can be shown that the $2r^{th}$-order cumulants of the output of the system satisfy

$$\sum_{i=n_0}^{n_o+m-1} C_{l_1 l_2 \cdots l_{2r-1} i} \cdot y_{i-n_0} = C_{l_1 l_2 \cdots l_{2r-1} l_{n_o+m}} \tag{5}$$

where

$$y_i = (-1)^{m-1-i} \sum_{j=1}^{C_{m-i}^{m}} \left\{ \prod_{k=1}^{m-i} a_{l_{jk}} \right\}, \; i = 0, 1, \cdots, m-1 \tag{6}$$

and $r \ge 2$ and $l_{jk} \in \{1, 2, \cdots, m\}$.

From the definition of y_i, when $m = 2$, we have $y_0 = -a_1 a_2$ and $y_1 = a_1 + a_2$, and when $m = 3$, $y_0 = a_1 a_2 a_3$, $y_1 = -(a_1 a_2 + a_1 a_3 + a_2 a_3)$ and $y_2 = a_1 + a_2 + a_3$. Therefore, the output cumulants are related by y_i and a system of linear equation can then be formulated. From the definition of y_i, a_i can be solved by putting y_i as the coefficients of an equation whose roots are essentially a_i.

Denote $\hat{(\cdot)}$ as the estimated value of $(\cdot)$, the algorithm can be formulated as follows.

1. Estimate the $2r^{th}$ order cumulants of the array output from N snapshots.

2. Obtain a set of linear equations using (5).

3. Express the linear equation in matrix form, we have

$$\hat{\mathbf{Q}}\mathbf{Y} = \hat{\mathbf{C}} \tag{7}$$

where

$$\hat{\mathbf{Q}} = \begin{bmatrix} \hat{C}_{1\cdots 11} & \hat{C}_{1\cdots 12} & \cdots \hat{C}_{1\cdots 1m} \\ \hat{C}_{1\cdots 12} & \hat{C}_{1\cdots 13} & \cdots \hat{C}_{1\cdots 1,m+1} \\ \cdots & \cdots & \cdots \\ \hat{C}_{1\cdots 1,n-m} & \hat{C}_{1\cdots 1,n-m+1} & \cdots \hat{C}_{1\cdots 1,n-1} \\ \hat{C}_{1\cdots 121} & \hat{C}_{1\cdots 122} & \cdots \hat{C}_{1\cdots 12,m} \\ \cdots & \cdots & \cdots \\ \hat{C}_{1\cdots 12,n-m} & \hat{C}_{1\cdots 12,n-m+1} & \cdots \hat{C}_{1\cdots 12,n-1} \\ \cdots & \cdots & \cdots \\ \hat{C}_{n\cdots n1} & \hat{C}_{n\cdots n2} & \cdots \hat{C}_{n\cdots nm} \\ \cdots & \cdots & \cdots \\ \hat{C}_{n\cdots n,n-m} & \hat{C}_{n\cdots n,n-m+1} & \cdots \hat{C}_{n\cdots n,n-1} \end{bmatrix} \tag{8}$$

and

$$\hat{\mathbf{C}} = \begin{bmatrix} \hat{C}_{11\cdots 1,m+1} \\ \hat{C}_{11\cdots 1,m+2} \\ \cdots \\ \hat{C}_{1\cdots 11,n} \\ \hat{C}_{1\cdots 12,m+1} \\ \cdots \\ \hat{C}_{1\cdots 12,n} \\ \cdots \\ \hat{C}_{n\cdots n1,m+1} \\ \cdots \\ \hat{C}_{n\cdots nn} \end{bmatrix} \tag{9}$$

whilst $\mathbf{Y} = [y_0 \; y_1 \; \cdots \; y_{m-1}]^T$ and y_i are given by (6). Obtain $\mathbf{Y}$ from

$$\mathbf{Y} = \hat{\mathbf{Q}}^{\dagger}\hat{\mathbf{C}} \tag{10}$$

where $\hat{\mathbf{Q}}^{\dagger}$ is the pseudoinverse of $\hat{\mathbf{Q}}$.

4. Solve the equation whose coefficients are the elements of $\mathbf{Y}$, i.e.

$$a^m - y_{m-1}a^{m-1} - \cdots y_1 a - y_0 = 0 \quad (11)$$

The DOA of the system is then given by the roots of (11).

2.4 Discussion

There are a few issues about the algorithm that need to be addressed. First, it has not been mentioned how the number of impinging sources is determined. The number of sources can be estimated by information theoretical criteria such as AIC[9] or MDL[10]. Shamsunder[11] derived a cumulant-based source number detection method. In fact, the rank of the output cumulant matrix can also provide an estimate of m. The output cumulant matrix is defined in a similar way as the output covariance matrix. In the simulation below, we use the rank to predict the number of sources.

Second, the computation requirement of the proposed algorithm, apart from estimating the output cumulant, requires singular value decomposition and root solving. The first operation calculates the pseudoinverse of $\hat{\mathbf{Q}}$ which in turn is used to evaluate the coefficient vector $\mathbf{Y}$. The DOA is then estimated by solving the equation formed by the coefficient vector $\mathbf{Y}$. By deliberately selecting or partitioning the $\mathbf{Q}$ matrix into square matrices, the pseudoinverse is therefore replaced by the ordinary inverse of a matrix which is much easier and computationally more efficient to be handled.

Last, if the signal sources are independent, (5) can be rewritten as

$$\sum_{i=n_o}^{n_o+m-1} C_L \cdot y_{i-n_o} = C_{L+n_o+m} \quad (12)$$

where L is the sum of the indices of the output cumulant multiplied by y_0. An example of such a generalized set of equation can be obtained by fixing the first $(2r-1)^{th}$ indices but varying the $2r^{th}$ index only. By so doing, estimation of DOA for a system with equal number of sensors and sources or a less-sensor-more-source system is also possible.

3 SIMULATION RESULTS

Simulations are performed to validate the algorithm. The first simulation shows the performance of the algorithm in two aspects, firstly when additive white Gaussian noise is present and secondly for different number of snapshots. With the aforesaid assumptions, 2 uniformly distributed signals with equal power impinging on 4 uniformly spaced sensors at DOAs -10 deg and -10 deg respectively were used. The estimation result of 30 Monte Carlo trials using different number of snapshots under various SNRs are tabulated below.

	Number of Snapshots			
SNR	200	400	1000	2000
∞	-10.00, 10.00	-10.00, 10.00	-10.00, 10.00	-10.00, 10.00
20	-9.98, 9.98	-9.95, 9.95	-9.96, 9.94	-9.94 9.96
10	-10.07, 10.05	-9.98, 10.03	-9.95, 9.94	-9.98, 9.94
5	-10.47, 10.57	-10.32, 10.29	-10.10, 10.03	-10.07 10.06
0	-11.47, 13.68	-8.93, 13.23	-11.33, 10.45	-10.61, 10.73

Table 1: Results for the proposed algorithm under white Gaussian noise of a 4-sensor-2-source system

Table.1 showed that the performance of the algorithm is satisfactory under different SNRs. The estimated DOAs are accurate up to 2 significant digits when 1000 snapshots are used for SNR greater than 5dB. Without the corrupting noise, the algorithm gives exactly the true DOAs when the number of snapshots vary from 200 to 2000. It is seen that the number of snapshots used does not affect the performance of the algorithm significantly at high SNRs. The deviation of the estimates from the true values, as expected, increases as the SNR decreases. However, with sufficient snapshots, accurate results can still be obtained under very low SNR.

Next, we test the algorithm when colored noise is presented. A system of 4 sensors and 2 sources is simulated. The sources are uniformly distributed and with equal power. The additive noise is colored Gaussian noise generated by some auto-regressive processes. The sources arrived at the sensor array at -10 deg and 15 deg respectively. The simulation results of 30 Monte Carlo runs are tabulated in Table.2.

	Number of Snapshots			
SNR	200	400	1000	2000
20	-9.95, 14.84	-9.95, 14.87	-9.94, 14.84	-9.96, 14.83
10	-10.01, 14.96	-9.97, 14.77	-9.97, 14.81	-9.95, 14.78
5	-10.80, 15.60	-10.37, 15.23	-10.23, 15.03	-10.06, 14.81
0	-15.35, 19.15	-13.89, 16.79	-11.40, 16.47	-10.81, 15.70

Table 2: Results for the proposed algorithm under colored Gaussian noise of a 4-sensor-2-sources system

Accurate simulation results are obtained even at fairly low SNRs. With 2000 snapshots, the algorithm still works pretty good when the SNR is as low as 0dB. Furthermore, it is seen that if SNR is sufficiently high, the estimates seems to be less dependent on the number of snapshots used. From the above tables, the performance of the algorithm is comparable under both white noise and color noise and as such the theoretical prediction is verified.

The last experiment revealed the performance of the proposed algorithm when the number of sources equals to that of sensors. It is worth to mention that such cases cannot be estimated by traditional covariance based methods[1-3]. A 2-sensor-2-source system is simulated. Two independent uniformly distributed sources with equal power impinging at the 2 sensors at -10 deg and 15 deg were used. The output of the sensor was corrupted by additive color Gaussian noise. The **Q** matrix and the **C** vector are defined by

$$\mathbf{Q} = \begin{bmatrix} C_{1111} & C_{1112} \\ C_{1211} & C_{1212} \\ C_{2211} & C_{2212} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} C_{1122} \\ C_{1222} \\ C_{2222} \end{bmatrix}$$

The simulation results for 30 Monte Carlo trials are shown in Table.3.

	Number of Snapshots			
SNR	200	400	1000	2000
20	-10.38, 14.91	-10.04, 15.05	-10.18, 15.08	-9.84, 14.77
10	-10.48, 14.87	-10.91, 15.61	-9.88, 14.76	-10.03, 14.85
5	-11.91, 15.84	-10.13, 14.52	-9.79, 15.25	-10.75, 14.68
0	-23.63, 21.48	-16.98, 18.64	-13.90, 16.41	-9.25 17.21

Table 3: Results for the proposed algorithm under colored Gaussian noise of a 2-sensor-2-source system

The algorithm works fairly well and accurate estimates are obtained. Although the performance degrades rapidly as compared to that of the 4 sensors system, this experiment demonstrates the potential of the proposed algorithm to estimate unknown sources using less sensors.

4 CONCLUSION

An DOA estimation algorithm based on higher-order cumulant is proposed. The algorithm makes use of the higher-order cumulants of the output of the sensor array. By arranging the cumulants in a specific matrix form, the nonlinear optimization required by other higher-order cumulant based DOA estimation algorithms is eliminated which makes it more computationally efficient. Simulation reveals that the algorithm can estimate DOAs accurately under noisy environment.

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