STATIONARY MOMENTS OF A POLYNOMIAL PHASE SIGNAL,APPLICATION TO PARAMETER ESTIMATION

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ABSTRACT

This communication addresses the problem of estimating the parameters of a polynomial phase signal using an original approach: although this signal is clearly non stationary, some of its high order moments are shift invariant. The condition verified by the delays of these "stationary" moments is derived in the noiseless and noisy case. It is demonstrate that the only identifiable phase parameter is the highest order coefficient, the estimation requiring moments of order at least the double of the phase degree. An algorithm relying on these high order moments is derived and its performances are presented and compared to a recent algorithm.

1 INTRODUCTION

This article addresses the estimation of the parameters of a polynomial phase signal. This problem, encountered in different radar systems, is usually solved using a time frequency analysis or phase-only algorithms. For a detailed introduction on this topic see for example [4]. This paper investigates a parametric solution based on statistical properties of this signal: although it is clearly non-stationary, some of its high order moments are shift invariant. This property is exploited for the retrieval of the phase parameters.

2 POLYNOMIAL PHASE SIGNALS HIGH ORDER MOMENTS

2.1 The noiseless case

We define a noisy polynomial phase signal y_n as:

$$x_n = A\exp\{j\sum_{q=0}^{M} a_q n^q\}, \quad y_n = x_n + e_n, \qquad (1)$$

where a_0 is a random variable uniformly distributed in the interval $[0, 2\pi)$ and e_n is a complex independent identically distributed (iid) noise. The $2p$-th order moment of the signal z_n is defined as:

$$M_{2p,z}(n; l_1, \ldots, l_{2p}) \triangleq E\left\{\prod_{k=1}^{p} z_{n+l_k} \prod_{k=p+1}^{2p} z^*_{n+l_k}\right\}. \qquad (2)$$

z_n is said $(l_1, \ldots, l_{2p})$-stationary if $M_{2p,z}(n; l_1, \ldots, l_{2p})$ is shift invariant, i.e., $M_{2p,z}(n; l_1, \ldots, l_{2p})$ is not a function of n. In this case, the index n will be omitted in $M_{2p,z}(n; l_1, \ldots, l_{2p})$. It is worthy to note that the definitions obtained with a different number of conjugated and unconjugated terms lead to a $2p$-th order moment of x_n that equals zero.

Proposition 1 *The signal x_n in (1) is $(l_1, \ldots, l_{2p})$-stationary if the following condition, referred as $H(p, M)$, holds:*

$$H(p, M) : \forall q \in \{1, \ldots, M-1\}, \sum_{k=1}^{p}(l_k^q - l_{k+p}^q) = 0. \qquad (3)$$

In this case the stationary moments equal:

$$M_{2p,x}(l_1, \ldots, l_{2p}) = A^{2p}\exp\{ja_M \sum_{k=1}^{p}(l_k^M - l_{k+p}^M)\}. \qquad (4)$$

Proof: The $2p$-th order moments $M_{2p,x}(n; l_1, \ldots, l_{2p})$ equal $A^{2p}\exp\{j\phi\}$ with:

$$\begin{aligned}
\phi &= \sum_{k=1}^{p}\sum_{q=1}^{M} a_q((n+l_k)^q - (n+l_{k+p})^q) \\
&= \sum_{k=1}^{p}\sum_{q=1}^{M} a_q \sum_{m=0}^{q}\binom{q}{m}(l_k^{q-m} - l_{k+p}^{q-m})n^m \\
&= \sum_{m=0}^{M}\underbrace{\left(\sum_{q=\max(1,m)}^{M} a_q \binom{q}{m}\sum_{k=1}^{p}(l_k^{q-m} - l_{k+p}^{q-m})\right)}_{b_m} n^m.
\end{aligned}$$

This polynomial of the variable n is not a function of n if $\forall m = 1 \ldots M$, $b_m = 0$. $b_1 = 0$ implies $\sum_{q=1}^{M} a_q \binom{q}{1}\sum_{k=1}^{p}(l_k^{q-1} - l_{k+p}^{q-1}) = 0$. This equation has to be verified $\forall a_q$. This implies $H(p, M)$ that implies that the others b_m also equal 0.

The coefficient b_0 equals $\sum_{q=0}^{M} a_q \sum_{k=1}^{p}(l_k^q - l_{k+p}^q)$. Application of $H(p, M)$ to this equality terminates the proof of equation (4). ■

The only identifiable parameter of a polynomial phase signal from its stationary moments is the coefficient of highest degree, a_M. The estimation of all the a_k can be performed using the iterative algorithm consisting in: estimate a_M, multiply the signal by $\exp\{-ja_M n^M\}$, estimate a_{M-1},...

The identifiability of the parameter a_M is precised by the following proposition:

Proposition 2 *The coefficient a_M of the signal in (1) is only identifiable from the stationary moments of order higher or equal to $2M$. The stationary moments of order $2M$ of x_n equal:*

$$M_{2M,x}(0,l_2,\ldots,l_{2M}) = A^{2M}\exp\{(-1)^M jMa_M \prod_{k=1}^{M} l_{M+k})\} \quad (5)$$

Proof : Consider the two $p\times p$ diagonal matrices $C = Diag\{l_1,\ldots,l_p\}$, $D = Diag\{l_{p+1},\ldots,l_{2p}\}$. The condition $H(p,M)$ can be rewritten in term of these matrices as:

$$H(p,M): \forall q \in \{1,\ldots,M-1\}, tr\{C^q\} = tr\{D^q\}, \quad (6)$$

where $tr\{M\}$ denotes the trace of the matrix M. If we apply the Hamilton-Cayley theorem, [2], to these matrices, we obtain:

$$C^p - \sum_{q=1}^{p} c_q C^{p-q} = 0, \quad D^p - \sum_{q=1}^{p} d_q D^{p-q} = 0, \quad (7)$$

where c_k and d_k are the coefficients of the characteristic polynomial of C and D. The Faddeev method, [2], allows the recursive computation of these coefficients using the relations:

$$kc_k = tr\{C^k\} - \sum_{q=1}^{k-1} c_q tr\{C^{k-q}\}, k = 1\ldots p \quad (8)$$

$$kd_k = tr\{D^k\} - \sum_{q=1}^{k-1} d_q tr\{D^{k-q}\}, k = 1\ldots p \quad (9)$$

with $c_0 = 1$ and $d_0 = 1$. The recursive application of $H(p,M)$ to these equations gives:

$$\forall q \in \{1,\ldots,\min(M-1,p)\}, \quad c_q = d_q. \quad (10)$$

Consider first the case $p < M$. The trace of the difference of the two equalities (7) together with (10) and $H(p,M)$ gives $tr\{C^p\} = tr\{D^p\}$. If we now multiply equations (7) by C and D, the trace of the difference of these two equalities will give $tr\{C^{p+1}\} = tr\{D^{p+1}\}$. Iterating, we then obtain $\forall q$, $tr\{C^q\} = tr\{D^q\}$. The application of this result with $q = M$ to (4) terminates the proof of the first assertion of the proposition.

Consider now the case $p = M$. In this case, the trace of the difference of the two equalities (7) together with (10) and $H(M,M)$ gives

$$tr\{C^M\} - tr\{D^M\} = (c_M - d_M)tr\{I\} \quad (11)$$

$$= M.((-1)^{M-1}\prod_{k=1}^{M} l_k - (-1)^{M-1}\prod_{k=1}^{M} l_{p+k}) \quad (12)$$

Application of this result with $l_1 = 0$ to (4) terminates the proof. ∎

2.2 The noisy case

In the noisy case, we will consider the moments of the signal y_n at lags for which x_n is $(l_1,\ldots,l_{2p})$-stationary. For brevity, we introduce the ordered k-uple $\mathcal{L}_k$ constituted by a subset of k elements of $(l_1,\ldots,l_p)$. $\mathcal{L}_k^*$ is a k-uple of $(l_{p+1},\ldots,l_{2p})$, $\overline{\mathcal{L}}_k$ the complement of $\mathcal{L}_k$ in $(l_1,\ldots,l_p)$ and $\overline{\mathcal{L}}_k^*$ the complement of $\mathcal{L}_k^*$ in $(l_{p+1},\ldots,l_{2p})$. Consequently, we will use the condensated notation $M_{2k,z}(n;\mathcal{L}_k,\mathcal{L}_k^*)$ for the $2k$-th order moment of z_n. Using the circularity of e_n and the hypothesis on a_0 distribution, we can expand the $2p$-th order moment of y_n as:

$$M_{2p,y}(\mathcal{L}_p,\mathcal{L}_p^*) = \sum_{\mathcal{L}_k,\mathcal{L}_k^*} M_{2k,x}(n;\mathcal{L}_k,\mathcal{L}_k^*)M_{2(p-k),e}(\overline{\mathcal{L}}_k,\overline{\mathcal{L}}_k^*). \quad (13)$$

Unfortunately, the $(l_1,\ldots,l_{2p})$-stationarity of x_n does not imply the $(\mathcal{L}_k,\mathcal{L}_k^*)$-stationarity of x_n and then the $(l_1,\ldots,l_{2p})$-stationarity of y_n. However, (13) suggests the following proposition:

Proposition 3 *For all solutions of $H(p,M)$ verifying $\{l_1,\ldots,l_p\}\cap\{l_{p+1},\ldots,\ l_{2p}\} = \emptyset$ we have $M_{2p,y}(l_1,\ldots,l_{2p}) = M_{2p,x}(l_1,\ldots,l_{2p})$. If moreover $p = M$:*

$$M_{2M,y}(0,l_2,\ldots,l_{2M}) \quad A^{2M}\exp\{(-1)^M jMa_M\prod_{k=1}^{M} l_{M+k}\} \quad (14)$$

Proof: This result is a direct consequence of (13) and proposition 2 ∎

2.3 The set of stationary moments delays

We will limit our study to the minimal order moment: $p = M$. According to the previous sections, we need to find the $2M$-uple verifying: $H(M,M)$, $\{l_1,\ldots,l_M\}\cap\{l_{M+1},\ldots,l_{2M}\} = \emptyset$. We suggest to find them using $2M-1$ loops to test condition $H(M,M)$. In this context, it is important in order to reduce the computational cost of the scan to define a set of minimal size containing the l_k. A reasoning similar to [6] leads to the following ordered set:

$$l_1 = 0, 0 \le l_2 \le \cdots \le l_M, 1 \le l_{M+1} \le \cdots \le l_{2M}. \quad (15)$$

An important remark is that if $(l_1,l_2,\ldots,l_{2M})$ verifies $H(M,M)$ and $\{l_1,\ldots,l_M\}\cap\{l_{M+1},\ldots,l_{2M}\} = \emptyset$, $\forall q \neq$

$0, \{ql_1, ql_2, \ldots, ql_{2M}\}$ also verifies these two conditions. A direct consequence is that the set of interest can be reduced to only its coprime members. An element of this set will be called a root of order M. Table (1) gives the 11 first roots of order 3.

3 SAMPLE ESTIMATES STATISTICAL PROPERTIES

The stationary moments of the signal under scope may be estimated on the basis of a single realizations of the time series y_n, $n = 1 \ldots N$. The sample moment function can be estimated as:

$$M^N_{2p,y}(\mathcal{L}_p, \mathcal{L}^*_p) = \frac{1}{N_s} \sum_{n=1-l_m}^{N-l_M} \prod_{l\in\mathcal{L}_p, m\in\mathcal{L}^*_p} y_{n+l} y^*_{n+m}, \quad (16)$$

where $l_M = \max\{\mathcal{L}_p \cup \mathcal{L}^*_p\}$, $l_m = \min\{\mathcal{L}_p \cup \mathcal{L}^*_p\}$ and $N_s = N - l_M + l_m$.

The statistical properties of these estimates is a fundamental problem. It is straightforward that the sample moment function (16) is an unbiased estimator of $M_{2p,y}(\mathcal{L}_p, \mathcal{L}^*_p)$. The next question is the convergence in probability of $M^N_{2p,x}(\mathcal{L}_p, \mathcal{L}^*_p)$ to $M_{2p,x}(\mathcal{L}_p, \mathcal{L}^*_p)$. The answer is given by the following proposition:

Proposition 4 *The sample estimate $M^N_{2p,x}(\mathcal{L}_p, \mathcal{L}^*_p)$ tends to $M_{2p,x}(\mathcal{L}_p, \mathcal{L}^*_p)$ in the mean square sense as N goes to infinity.*

Proof: See [1] for a demonstration. ■

4 Estimation algorithm

4.1 Mathematical derivation

The last point is the derivation of an algorithm for the estimation of the coefficient a_M of the signal (1). This estimation will rely on a matching between estimated stationary moments and their theoretical expression. Consider a root of order M: $(0, l_2, \ldots, l_{2M})$. The estimation procedure will exploit the moments associated to the first L multiples of this root. For purpose of notational simplification we use the following notation:

$$m(q) \stackrel{\Delta}{=} M_{2M,y}(0, ql_2, \ldots, ql_{2M}), \quad q = 1, \ldots, L. \quad (17)$$

Proposition 3, implies:

$$m(q) = A^{2M} \exp\{jq^M \alpha_M\}, \quad q = 1, \ldots, L, \quad (18)$$

where

$$\alpha_M = (-1)^M M a_M \prod_{k=1}^{M} l_{M+k}. \quad (19)$$

To avoid a function minimization, the algorithm will consist on a least squares fit between the angles of the estimated moment $\hat{m}(q)$, $q = 1, \ldots, L$ and their theoretical value. According to (18,19), this solution is:

$$\hat{a}_M = \frac{\sum_{q=1}^{L} q^M \text{angle}(\hat{m}(q))}{(-1)^M M \prod_{k=1}^{M} l_{M+k} \cdot \sum_{q=1}^{L} q^{2M}}.$$

The previous expression assumes phase unwrapping during the computation of the angles. To avoid this problem we will require that $|q^M \alpha_M| < \pi$, $q = 1, \ldots, L$. Thus, the condition required to avoid any ambiguity about a_M is:

$$|a_M| < \frac{\pi}{L^M M \prod_{k=1}^{M} l_{M+k}}. \quad (20)$$

For notational simplification, this algorithm will be referred in the sequel by SMF (stationary moments fitting).

4.2 Computer simulations

In order to evaluate the performances of SMF, computer simulations using an hyperbolic phase signal ($M = 3$) of $N = 20$ samples have been drawn. The phase coefficients are: $a_3 = 0.0005$, $a_2 = 0.007$, $a_1 = 0.003$ and $a_0 = 1$. The mean square error (MSE) of $\hat{a}_3$ has be estimated using 500 noise realization for each signal to noise ratio.

The first question is the determination of the parameter L. The first order 3 root $(3, 3, 1, 1, 4)$, has been selected and MSE curves associated to various L have been computed. If we arbitrarily assume that an average on at least $N/3$ samples is required in (16), the maximum value of L is given by the integer part of $2N/(3l_M - 3l_m)$. Figure (1) gives the results of these simulations: the MSE decreases as L increases. Consequently, the maximum value of L will be retained in the following simulations.

The next problem is the selection of the root. Figure (2) represents the MSE curves for the first three roots $(3, 3, 1, 1, 4)$, $(4, 5, 1, 2, 6)$, $(5, 7, 1, 3, 8)$ with respectively $L = 4$, 3 and 2. The root giving the lower MSE is $(4, 5, 1, 2, 6)$. This result illustrate the dual influence of the parameter L and the deviation $l_M - l_m$. The MSE decreases when both these quantities will increase. In this sense, the pair $(4, 5, 1, 2, 6)$ represents a compromise that will be retained in the sequel.

Finally, the performances of SMF have been compared to the Cramer Rao lower bound derived in [5] and to a parametric method for analysis of constant-amplitude polynomial phase signals: the Discrete Polynomial-Phase Transform (DPPT), [4]. The first step of this method is a transformation of the signal into a single tone signal. This is performed iteratively by $M - 1$ phase differentiations. At each step, the phase differentiation of the current signal is obtained multiplying the sample at instant n by the conjugated sample at instant $n - \tau$. The coefficient a_M is then related to the global maximizer of the transformed signal periodogram. The framework derived in this paper allows a novel interpretation of the DPPT. In particular, it can be easily demonstrated that the DPPT retrieves a_M from the global maximizer of the periodogram of estimated 2^M order moments of x_n.

l_2	l_3	l_4	l_5	l_6	$-3l_4l_5l_6$
3	3	1	1	4	-12
4	5	1	2	6	-36
5	7	1	3	8	-7
5	8	2	2	9	-108
6	9	1	4	10	-120
6	11	2	3	12	-216
7	7	1	4	9	-108
7	8	2	3	10	-180
7	11	1	5	12	-180
9	9	3	3	12	-324
9	10	1	6	12	-216

Table 1: 11 first roots of order 3.

Figure (3) gives the Monte-Carlo simulations results and the Cramer Rao lower bound. The DPPT has been implemented using an optimization algorithm initialized by the estimate given by an FFT of the transformed signal zero padded to 2^9. The parameter τ equals N/M.

For SNR>10dB, performances of the proposed algorithm are very close to the CRLB and performances of DPPT correspond to the values predicted by the theoretical analysis derived in [4]. For SNR<10dB, DPPT exhibits the traditional breakdown whereas the efficiency of SMF remains almost constant. This result could be explained by the fact that the moments order involved are respectively 2^M for DPPT and $2M$ for SMF, these last ones being estimated with better precision.

References

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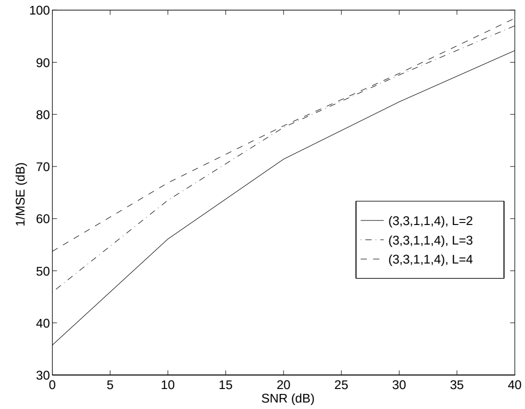

Figure 1: MSE for $\hat{a}_3$ using SMF. The root is $(3,3,1,1,4)$ and $L = 2, 3$ and 4.

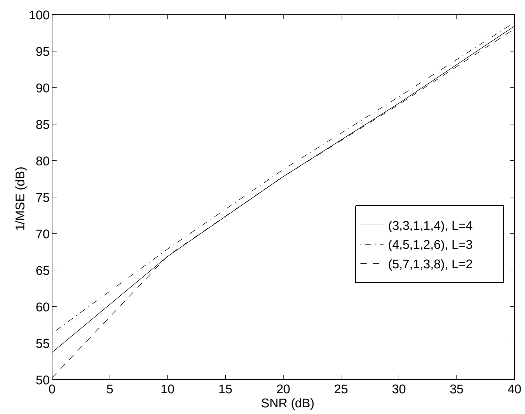

Figure 2: MSE for $\hat{a}_3$ using SMF with $(3,3,1,1,4)$ and $L = 4$, $(4,5,1,2,6)$ and $L = 3$, $(5,7,1,3,8)$ and $L = 2$.

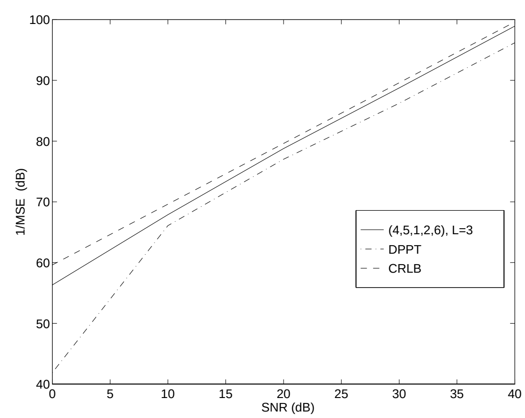

Figure 3: MSE for $\hat{a}_3$ using SMF, DPPT and Cramer Rao lower bound.