# $L_\infty$ BLIND DECONVOLUTION FOR THE GENERALIZED AR MODEL

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**ABSTRACT**

Applying the convex cost function $L_\infty$ to the blind deconvolution of general non-minimum phase AR(u) models is studied. A simple and realizable constraint is proposed for the $L_\infty$ deconvolution. With this constraint, except for a gain, the model parameter is the unique solution of the $L_\infty$ deconvolution. The strong consistency of the estimator of the model parameter defined by the sample version of $L_\infty$ norm is presented. An algorithm is suggested for the iterative computation of the estimator. Simulation examples show the proposed approach works well for apprepriate blind equalization problems.

## 1 INTRODUCTION

Blind deconvolution is one of the basic problems in signal processing and has a great number of applications in communication, geophysics, radar, ultrasonic technology, image restoration, etc.

The general model for blind deconvolution is

$$y_n = \sum_{k=-\infty}^{\infty} c_k x_{n-k}\,, \tag{1}$$

(i) $\{c_k\}$ is a stable deterministic system response, i.e. $\sum_{k=-\infty}^{\infty}|c_k| < \infty$, which, in communications, describes the intersymbol interference, (ii) the input signal $\{x_k\}$ is a zero-mean i.i.d. random sequence. Denote $\mathbf{c}=\{c_k,-\infty<k<\infty\}$. It is assumed that there exists $\mathbf{d}=\{d_k, \infty<k<\infty\}$,which is called the inverse filter of $\mathbf{c}$, such that $\mathbf{d}\otimes\mathbf{c}=\boldsymbol{\delta}$, where $\otimes$ means convolution and $\boldsymbol{\delta}$ is a delta sequence. In the most general case, $\{c_k\}$ and the probability distribution of $x_k$ are unknown. The blind deconvolution problem for (1) is to estimate both of the input sequence $\{x_n\}$, and the system response $\{c_k\}$.

Numerous approaches have been developed to solve the blind deconvolution problem. One approach is based on solving an optimization problem. The optimization approach for deconvolution can be described as the determination of certain cost function $f(\cdot)$ such that

$$\mathbf{d} = \{d_k, -\infty < k < \infty\} = \underset{\mathbf{h}\in H}{\text{Optimizer}}\; E[f(\sum_{k=-\infty}^{\infty} h_k y_{n-k})], \tag{2}$$

where $\mathbf{h}=\{h_k, -\infty<k<\infty\}$ and $H$ is defined by certain specified normalization constraint on $\mathbf{h}$.

With the additional condition $|x_n| \le M$, Lucky (1965) suggested using $f(\sum_{k=-\infty}^{\infty} h_k y_{n-k}) = \sup_{\omega\in\Omega}\left|\sum_{k=-\infty}^{\infty} h_k y_{n-k}\right| = M\|\mathbf{h}\otimes\mathbf{c}\|_{\ell_1}$ as the cost function in (2), where $\|\cdot\|_{\ell_1}$ denotes the $\ell_1$ norm . Recently, Vembu et al. (1994) stated that with the assumption of boundedness of $x_n$ in (1), except for some constant gain and shift, $\mathbf{d}$ is the solution of the minimization problem

$$\begin{cases} \min\limits_{\mathbf{h}\in\ell_1} \sup\limits_{\omega\in\Omega} \left|\sum_{k=-\infty}^{\infty} h_k y_{n-k}\right| = \min\limits_{\mathbf{h}\in\ell_1} M\|\mathbf{h}\otimes\mathbf{c}\|_{\ell_1}, \\ \text{subj. to } h_0 = 1. \end{cases} \tag{3}$$

The proof of this statement in that paper actually shows that if $d_0 = \max_k |d_k| = 1$, then $\mathbf{d}$ is the solution of (3). The constraint in (3) is simple and easily realized. But, unfortunately, the argument cannot ensure that except for some gain and shift, $\mathbf{d}$ is the unique solution of (3) when $\mathbf{d}$ has nonunique maximum components in absolute value. Since $\|\mathbf{h}\otimes\mathbf{c}\|_{\ell_1}$ is a convex, but not strictly convex function about $\mathbf{h}$, if the solution of (3) is not unique, then there are infinite number of solutions for (3). The differences between these solutions is not only confined to some gain or time shift. This means that solving (3) cannot uniquely determine $\mathbf{d}$ up to some gain and shift. The nonuniqueness of the solution of (3), at least in theory, decreases the validity of applying (3) to the blind deconvolution of (1).

In this paper, for simplicity of description, blind deconvolution of the generalized AR(u) model is considered. A simple and realizable constraint is proposed for the $L_\infty$ deconvolution of the generalized AR(u) model. In this case, $\mathbf{d}$ is the unique solution of $L_\infty$ deconvolution except for some gain. Furthermore, the estimator of $\mathbf{d}$, say $\hat{\mathbf{d}}$, is defined

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based on the sample version of the $L_\infty$ norm. The strong consistency of $\hat{\mathbf{d}}$ is presented. Moreover, an algorithm is proposed for seaching $\hat{\mathbf{d}}$. Some simulation results are also presented to show that the proposed algorithm works well.

## 2. GENERAL AR MODEL AND ITS BLIND DECONVOLUTION

A precise description of the generalized AR($u$) model without the assumption of minimum phase is presented below. Suppose $\{y_n\}$ is the stationary solution of the stochastic difference equation

$$d_0 y_n + d_1 y_{n-1} + \cdots + d_u y_{n-u} = x_n, \quad \forall\ n, \tag{4}$$

where $\{x_n\}$ is a sequence of zero-mean non-Gaussian i.i.d random variables defined on a probability space ($\Omega$, $F$, P), $d_0 \neq 0$, $d_p \neq 0$, and $D(z) = d_0 + d_1 z^{-1} + \cdots + d_u z^{-u}$ has no zeros on the unit circle. It is not difficult to prove that $y_n = \sum_{k=-\infty}^{\infty} c_k x_{n-k}$ is the unique stationary solution to (4), where

$$C(z) = \sum_{k=-\infty}^{\infty} c_k z^{-k} = \frac{1}{D(z)}. \tag{5}$$

Since $D(z)$ has no zeros on the unit circle, the convergence region of C(z) contains the unit circle. Thus, $\sum_{k=-\infty}^{\infty} |c_k| < \infty$. Furthermore, if D(z) is a polynomial with nonminimum phase, then, it is well known that D(z) can be decomposed as

$$D(z) = D_*(z) P(z), \tag{6}$$

where $D_*(z) = d_{*0} + d_{*1} z^{-1} + \cdots + d_{*u} z^{-u}$ is minimum phase and $P(z) = \sum_{k=0}^{\infty} p_k z^{-k}$ is an all-pass system, i.e. $|P(e^{j\omega})| = 1$. Therefore, (5) is equivalent to

$$d_{*0} y_n + d_{*1} y_{n-1} + \cdots + d_{*u} y_{n-u} = w_n, \quad \forall\ n, \tag{7}$$

where $w_n = \sum_{k=0}^{\infty} p_k x_{n+k}$ due to $\frac{1}{P(z)} = P(z^{-1}) = \sum_{k=0}^{\infty} p_k z^k$ and $\{w_n\}$ is a zero-mean uncorrelated sequence with the same variance as $\{x_k\}$. That is, (7) can be regarded as a classical AR model. Unfortunately, since $\{y_n\}$ is not Gaussian, the commonly-used methods cannot be used to estimate the order u in (7). A minimum phase version of the approach for the identification of non-Gaussian ARMA by Lii (1990) can be applied to the order estimation of (7). Clearly, from $d_0 \neq 0$ and $d_u \neq 0$, we have $d_{*0} \neq 0$ and $d_{*u} \neq 0$. Therefore, the order of (7) is the same as that of (4). Hereafter, u is assumed to be known .

The blind deconvolution problem for the AR(u) model (4), now, is the estimation of $d_0, d_1, \cdots, d_u$ and the restoration of $\{x_n\}$ based on observations $y_0, \cdots, y_N$. In general, there are no other assumptions on the statistical properties of $\{x_n\}$ except that $\{x_n\}$ is non-Gaussian i.i.d with zero-mean. For the blind deconvolution of the generalized AR(u) model, the minimization approach with convex cost functions is to determine a certain convex function f(.) such that

$$\mathbf{d} = (d_0, d_1, \cdots, d_u)^T = \underset{\mathbf{h} \in H}{\text{minimizer}}\ E[f(\sum_{k=0}^{u} h_k y_{n-k})], \tag{8}$$

where $H$ is some set of $\mathbf{h} = (h_0, h_1, \cdots, h_u)^T$ with specified normalization constraint, such as $d_0 = 1$, $\max_{o \le i \le u} |d_i| = 1$, $\sum_{i=0}^{u} |d_i|^2 = 1$, or others.

## 3. THE UNIQUENESS OF A CLASS OF $L_\infty$ BLIND DECONVOLOTIONS

In fact, the $L_\infty$ cost function has been used to define the prediction error estimate (Ljung (1987). In this case, the constraint is $h_0 = 1$. For the minimum phase case, the solution of (8) with the $L_\infty$ cost function is unique. But, for the non-minimum phase case, the solution may be not unique. The following theorem shows that for the generalized AR(u) model, except for some gain, **d** is the unique solution of $L_\infty$ deconvolution with the constraint $\max_{0 \le i \le u} h_i = \max_{0 \le i \le u} |h_i| = 1$. this constraint is very simple but not convex. Let $gD(z) = h_0^* + h_1^* z^{-1} + \cdots + h_u^* z^{-u}$, where $g = 1/d_L$ and $|d_L| = \max_{0 \le i \le u} |d_i|$, and denote $\mathbf{h}^* = (h_0^*, \cdots, h_u^*)^T$.

**Theorem 1.** For the generalized AR(u) model with the assumption that $|x_n| \le M$, $\mathbf{h}^* = (h_0^*, \cdots, h_u^*)^T$ is the unique solution of the minimization problem

$$\begin{cases} \min_{\mathbf{h}=(h_0,\cdots,h_u)^T} \sup_{\omega \in \Omega} \left| \sum_{k=0}^{u} h_k y_{n-k} \right| \\ = \min_{\mathbf{h}=(h_0,\cdots,h_u)^T} M \sum_{n=-\infty}^{\infty} \left| \sum_{k=0}^{u} h_k c_{n-k} \right|, \\ \text{subj. to } \max_{0 \le i \le u} h_i = \max_{0 \le i \le u} |h_i| = 1. \end{cases} \tag{9}$$

Proof:

(i) Let

$$gD(z) = h_0^* + h_1^* z^{-1} + \cdots + h_u^* z^{-u}, \tag{10}$$

where g=1/$d_L$ and $\left|d_L\right| = \max_{0\le i\le u}\left|d_i\right|$. Therefore, $h_L^* = 1$ and $\left|h_i^*\right| \le 1,\ 0 \le i \le u$.

(ii) For a sequence **h**={$h_k$, -∞<k<∞}, if there exist $n_1$≤$n_2$ such that $h_{n_1} \ne 0,\ h_{n_2} \ne 0$ and $h_k = 0$ for k<$n_1$ or k>$n_2$, then we define $L$(**h**)=$n_2$-$n_1$ +1 which is the length of **h**.

(iii) Clearly,

$$\sum_{n=-\infty}^{\infty}\left|\sum_{k=0}^{u} h_k^* c_{n-k}\right| = |g| = 1/\max_{0\le i\le u}|d_i|. \qquad (11)$$

Define $h_k^{**} = h_{k+L}^*,\ k = 0, \pm 1, \cdots$. Then we have

$$h_0^{**} = 1,\ L(\mathbf{h}^{**}) = u+1,\ \left|h_k^{**}\right| \le 1,\ \forall k, \qquad (12)$$

and

$$\sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} h_k^{**} c_{n-k}\right| = |g| = 1/\max_{0\le i\le u}|d_i|. \qquad (13)$$

Obviously,

$$\min_{\{(h_0,\cdots,h_u)^T;\ \max_{0\le i\le u} h_i = \max_{0\le i\le u}|h_i|=1\}} M \sum_{n=-\infty}^{\infty}\left|\sum_{k=0}^{u} h_k c_{n-k}\right|$$

$$= \min_{\{\mathbf{h};\ L(\mathbf{h})=u+1,\ \max_{-\infty<i<\infty} h_i = \max_{-\infty<i<\infty}|h_i|=1\}} M \sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right|$$

$$\ge \min_{\{\mathbf{h};\ L(\mathbf{h})=u+1,\ h_0=1\}} M \sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right|. \qquad (14)$$

We now prove for any $\mathbf{h} \in \{\mathbf{h};\ L(\mathbf{h}) = u+1, h_0 = 1\}$, when **h** is neither a shift of $\mathbf{h}^{**}$ nor $\beta\mathbf{h}^{**}$, where β is a nonzero constant,

$$\sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right| > |g| = 1/\max_{0\le i\le u}|d_i|. \qquad (15)$$

Following Vembu, et al. (1994), it holds that

$$\sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right| - |g|$$

$$\ge \mathrm{sgn}(g) g \sum_{k=-\infty}^{\infty}(h_k - h_k^{**})\delta_k = |g|(h_0 - h_0^{**}) = 0. \qquad (16)$$

From (16) we know that if we assume that for some $\mathbf{h} \in$ {**h**; $L(\mathbf{h}) = u+1, h_0 = 1\}$, $\sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right| = |g| = \dfrac{1}{\max_{0\le i\le u}|d_i|}$, then for any n≠-L,

$$\left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right| = \left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right|\left|h_{-L-n}^{**}\right|. \qquad (17)$$

This means that either

$$\left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right| = 0 \qquad (18)$$

or

$$\sum_{k=-\infty}^{\infty} h_k c_{n-k} \ne 0 \text{ and } \left|h_{-L-n}^{**}\right| = 1. \qquad (19)$$

Since $L(\mathbf{h}^{**})$=u+1, it is asserted that at most for n=-L, ⋯, u-L, $\sum_{k=-\infty}^{\infty} h_k c_{n-k} \ne 0$. Denote

$$q_n = \sum_{k=-\infty}^{\infty} h_k c_{n-k}. \qquad (20)$$

It follows that

$$H(z)C(z) = q_{-L}z^L + \cdots + q_{u-L}z^{L-u} = z^L(q_{-L} + \cdots + q_{u-L}z^{-u}). \qquad (21)$$

Since C(z)=1/D(z), it is obtained that

$$z^{-L}H(z) = D(z)(q_{-L} + \cdots + q_{u-L}z^{-u}). \qquad (22)$$

Furthermore, because $d_0$≠0 and $d_u$≠0, in order that $L$(**h**) =u+1 and $h_0$=1, only one of $q_{-L}, \cdots, q_{u-L}$ can be nonzero. This means that except for some gain and shift, **d** is the unique solution of the minimization problem

$$\begin{cases} \min_{\mathbf{h}} \sup_{\omega\in\Omega}\left|\sum_{k=-\infty}^{\infty} h_k y_{n-k}\right| = \min_{\mathbf{h}} M \sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right| \\ \text{subj. to } L(\mathbf{h}) = u+1 \text{ and } h_0 = 1. \end{cases} \qquad (23)$$

On the other hand, for any $(\tilde{h}_0, \cdots, \tilde{h}_u)^T \in \{(h_0, \cdots, h_u)^T;\ \max_{0\le i\le u} h_i = \max_{0\le i\le u}|h_i| = 1\}$, by time shift such $\mathbf{h} \in \{\mathbf{h};\ L(\mathbf{h}) = u+1$ and $h_0 = 1\}$ can be found that $\sum_{n=-\infty}^{\infty}\left|\sum_{k=0}^{u} \tilde{h}_k c_{n-k}\right| = \sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} h_k c_{n-k}\right|$.

Therefore, from the uniqueness of (23) it is asserted that $\mathbf{h}^*$ is the unique solution of the minimization problem (9).

## 4. SAMPLE VERSION OF $L_\infty$ BLIND DECONVOLUTION

Suppose the samples $y_0, y_1, \cdots, y_N$ (N>u) are observed. The sample version of (9) based on these samples is

$$\begin{cases} \min_{(h_0,\cdots,h_u)} W^N(h_0, h_1, \cdots, h_u), \\ \text{subj. to } \max_{0\le i\le u} h_i = \max_{0\le i\le u}|h_i| = 1, \end{cases} \qquad (24)$$

where

$$W^N(h_0, h_1, \cdots, h_u) = \max\{\left|\sum_{k=0}^{u} h_k y_{p-k}\right|, \cdots, \left|\sum_{k=0}^{u} h_k y_{N-k}\right|\}. \qquad (25)$$

Denoting $\sup_{\omega\in\Omega}\left|\sum_{k=0}^{p} h_k y_{n-k}\right|$ by $W(h_0,\cdots,h_p)$, it can be shown that for any $(h_0,\cdots,h_u)$,

$$W^N(h_0,h_1,\cdots,h_u)\rightarrow W(h_0,\cdots,h_u),\ \text{w.p.1.} \quad (26)$$

Furthermore, denote this estimator by $\hat{\mathbf{h}}^* = (\hat{h}_0^*,\cdots,\hat{h}_u^*)^T$. Since $H$ in (9) and (24) is a bounded and closed set in a u+1-dimensional space with norm $\sum_{k=0}^{u}|h_k|$, $\underline{\kappa}$ is compact. From the compactness of $H$ and the uniqueness of the solution of (9) on $H$, we state the strong consistency of this estimator in the following theorem.

**Theorem 4.** $\left\|\hat{\mathbf{h}}^* - \mathbf{h}^*\right\|_{\ell_1} = \sum_{k=0}^{u}\left|\hat{h}_k^* - h_k^*\right| \rightarrow 0$, w.p.1.

## 4. ITERATIVE AGORITHM FOR $L_\infty$ BLIND DECONVOLUTION AND SIMULATION EXAMPLES

$L_\infty$ blind deconvolution can be implemented by solving, iteratively, the minimization problem (24) for the data $\{y_n\}$ in a sliding window. Many iterative algorithms, such as the simplex algorithm, conjugate direction method and its revised versions can be adopted. These methods need not calculate the derivative of cost functions for determining the direction of descent. the derivative of the cost function of (24) cannot be derived in a close form. Therefore, these methods are suitable to (24) in particular. In the following simulations, one version of the revised Powell's method (William(1994)) is used . The input is an i.i.d sequence with $P\{x_n=1\}=P\{x_n=-1\}=0.5$. $D(z)=1-1.5z^{-1}+0.36z^{-2} = (1-0.3z^{-1})(1-1.2z^{-1})$. This is a non-minimum phase AR(2) model with $d_0=1$, $d_1=-1.5$, and $d_2=0.36$. N is the size of the sample of $y_n$. The iteration is terminated when the difference of the cost function of two consecutive iterations is less than $\varepsilon=10^{-10}$. Denote the initial value and the value of $(\hat{h}_0,\hat{h}_1,\hat{h}_2)^T$ at the final iteration by $(\hat{h}_0^{(0)},\hat{h}_1^{(0)},\hat{h}_2^{(0)})^T$ and $(\hat{h}_0^{(f)},\hat{h}_1^{(f)},\hat{h}_2^{(f)})^T$. In the simulation, the number of iterations from $(\hat{h}_0^{(0)},\hat{h}_1^{(0)},\hat{h}_2^{(0)})^T$ to $(\hat{h}_0^{(f)},\hat{h}_1^{(f)},\hat{h}_2^{(f)})^T$ is about 7-10. All the results for the noiseless case are presented in Table 1 and Table 2. Then, we use the method to the noisy data. The noise is additive zero-mean Gaussian noise and the signal-to-noise ratio (SNR) equals 35.2 dB. The simulation results for this case is shown in Table 3. For all above cases, we use $\hat{H}(z)=\hat{h}_0^{(f)}+\hat{h}_1^{(f)}z^{-1}+\hat{h}_2^{(f)}z^{-2}$ as the equalizer . Applying the hard decision to the output of the equalizer we obtain the estimation of $\{x_n\}$, say $\{\hat{x}_n\}$. The probability of error is evaluated based on $\{x_n\}$ and $\{\hat{x}_n\}$. In all the above simulations, the probabilities of error are zero. Certainly, when N and SNR are small , the probability of error will be non-zero. The above results show that the proposed approach works well in the simulation examples.

**Table 1**

| $(\hat{h}_0^{(0)},\hat{h}_1^{(0)},\hat{h}_2^{(0)})^T=(1, 1, 1)^T$ | |
|---|---|
| N= 100 | $(\hat{h}_0^{(f)},\hat{h}_1^{(f)},\hat{h}_2^{(f)})^T = (1, -1.4976, 0.3570)^T$ |
| N= 500 | $(\hat{h}_0^{(f)},\hat{h}_1^{(f)},\hat{h}_2^{(f)})^T=(1, -1.4999, 0.3599)^T$ |

**Table 2.** N=500

| $\hat{h}_0^{(0)},\hat{h}_1^{(0)},\hat{h}_2^{(0)}$ | 0.33, 0.66, 1 | 1, 0.66, 0.33 |
|---|---|---|
| $\hat{h}_0^{(f)},\hat{h}_1^{(f)},\hat{h}_2^{(f)}$ | 1, -1.4999, 0.3599 | 1, -1.4999, 0.3599 |

| $\hat{h}_0^{(0)},\hat{h}_1^{(0)},\hat{h}_2^{(0)}$ | random (normal) | random (uniform) |
|---|---|---|
| $\hat{h}_0^{(f)},\hat{h}_1^{(f)},\hat{h}_2^{(f)}$ | 1, -1.4999, 0.3599 | 1, -1.4999, 0.3599 |

**Table 3.** N=500, SNR(signal to noise ratio)=35.2 dB

| $\hat{h}_0^{(0)},\hat{h}_1^{(0)},\hat{h}_2^{(0)}$ | 0.33, 0.66, 1 | random (normal) |
|---|---|---|
| $\hat{h}_0^{(f)},\hat{h}_1^{(f)},\hat{h}_2^{(f)}$ | 1, -1.4619, 0.3117 | 1, -1.4674, 0.3181 |

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