EXTENSION OF AUTOCOVARIANCE COEFFICIENTS SEQUENCE FOR PERIODICALLY CORRELATED RANDOM PROCESSES BY USING THE PARTIAL AUTOCORRELATION FUNCTION

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ABSTRACT

The extension of stationary process autocorrelation coefficients sequence is a classical problem in the field of spectral estimation. The periodically correlated (PC) processes have pratical importance and an interest according to their connection with stationary multivariate processes. That's why we propose a new approach to resolve the previous problem in this context. We use the partial autocorrelation function ($PACF$) of this processes class. The extension is so easy to describe. Next, we extend the maximum entropy method (MEM) to the degenerate case and show that the solution is given by a Periodic Autoregressive (PAR) process. Furthermore, the connection with the problem of multivariate stationary processes autocorrelation sequence is presented.

1 INTRODUCTION

Although the partial correlation notion was introduced by Yule [1], the one-to-one correspondence between autocovariance function (ACF) and $PACF$ for stationary processes is quite recent [2]. This result can be extended to the nonstationary case ([3], [4]). Thus the $PACF$ $\beta(\cdot,\cdot)$ constitutes another parametrization of the second order structure. But this function is easily identifiable by comparison with the classical ACF $R(\cdot,\cdot)$ which must be nonnegative definite.

The PC processes were introduced by Gladysev [5], as nonstationary processes with periodic ACF. These processes are meaningful according to their connection with stationary multivariate processes. At each PC process corresponds a stationary multivariate process and conversely [5]. The function $\beta(\cdot,\cdot)$ of such process is characterized by its periodicity likely to the function $R(\cdot,\cdot)$. The correspondence between these two functions is given by Levinson-Type Recursive algorithm of Sakai [6] or Pham [7] in the degenerate case. In [7] the coefficients $\beta(t,s)$ are triangular partial correlation matrices one's. In the stationary multivariate case, the choice in factorizing the covariance matrices of the prediction errors leads to different definitions of partial correlation matrices (see [8]). The triangular matrix results from the use of both lower-upper and upper-lower factorizations. Let note that the degenerate case is unusually for the stationary scalar processes but often occurs for the PC processes. So, it seems interesting to study this situation.

We suggest the following approach to state the extension problem: given the periodic function values $R(t,s)$ for $1 \le s \le t \le M$, one wish to determinate the conditions for these values to be the ACF one's and the way to describe the whole of such function extensions. The parametrization in terms of $PACF$ allows us to resolve this problem in an elegant way even in the degenerate case. Notice that the previous problem coincides with the extension of multivariate stationary process autocorrelation sequence one's when M is a multiple of the $R(\cdot,\cdot)$ function period. Otherwise it deals with a more general problem where the last autocorrelation matrix is not completely known.

Next, we are interested in the solution maximizing the entropy. In the stationary nondegenerate case (scalar or multivariate) the MEM was proposed by Burg [9]. The solution of this method is the one for which the innovation process variance (determinant of the covariance matrix, in multivariate case) is maximum. When the process is PC of period T, the innovation process variance, depending of time, is periodic of same period. Our approach to extend the MEM to the degenerate case is the following: the solution is the one for which the product of the innovation process variances not already null, is maximum. We constate that the above method coincides with the Burg's method for stationary multivariate regular processes and generalises the one resulting from the particular degenerate case treated by Inouye [10]. Our method is equivalent to put $\beta(t,s)=0$ everywhere this function is unknown. It follows that the solution is PAR. The algorithm providing the correspondence between the $R(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ functions, yields also this model parameters. The correspondence between the PAR processes class and the AR stationary multivariate processes one's is established by Pagano [11] and the relationship between the parameters of both models are quite easy.

Finally our approach is compared with the Alpay *et al.*'s one [12] which consider the extension problem in terms of cyclo-correlation functions.

Section 2 is devoted to the $PACF$ for PC processes and the extension problem is treated in Section 3.

2 PACF FOR PC PROCESSES

In this section we present the PC processes $PACF$ and give its correspondence with the ACF. Before, we need to recall some necessary results in the more general case of nonstationary processes ([3]).

2.1 Nonstationary General Case

Let $X(\cdot)=\{X(t), t\in \mathbb{Z}\}$ be a scalar complex valued nonstationary process with zero mean and ACF $R(\cdot,\cdot)$. We consider the Hilbert space $\overline{L}\{X(t), t\in\mathbb{Z}\}$ with the inner product $\langle U,V\rangle = E\left\{U\overline{V}\right\}$. So $R(\cdot,\cdot)$ is defined by $R(t,s) = \langle X(t), X(s)\rangle$, $(t,s)\in\mathbb{Z}^2$. We note $\varepsilon(t;s)$ the $(t-s)$-th order forward partial innovation. Putting $\varepsilon(t;t)=X(t)$ and $\sigma^2(t;s)=\|\varepsilon(t;s)\|^2$, the associated normalized innovation is defined for $s \leq t$ by $\eta(t;s) = \varepsilon(t;s)/\sigma(t;s)$ with the convention $0^{-1} = 0$. The backward innovations, obtained by reversing the time index, are denoted with a star, $\varepsilon^*(s;t)$ and $\eta^*(s;t) = \varepsilon^*(s;t)/\sigma^*(s;t)$ for $s \leq t$.Then [3] the $PACF$ $\beta(\cdot,\cdot)$ is defined on $\mathbb{Z}^2$ by

$$\beta(t,s) = \begin{cases} \langle \eta(t;s+1), \eta^*(s;t-1)\rangle & \text{if } s < t \\ \|X(t)\|^2 & \text{if } s = t \\ \overline{\beta(s,t)} & \text{if } s > t. \end{cases}$$

For $s < t$, $\beta(t,s)$ is the partial correlation coefficient between $X(t)$ and $X(s)$ in the set $\{X(s),\ldots,X(t)\}$. Putting $\beta(t,t) = Var\{X(t)\}$ instead of 1, the function $\beta(\cdot,\cdot)$, likely to the ACF, characterizes the second order properties of $X(\cdot)$ (see [3]). The advantage of this function is that it is easily identifiable: its magnetude is generally stictly less than 1, the equality to 1 translating the finite order singularities. Precisely, for $s < t$ $|\beta(t,s)| = 1$ if and only if s is the uppest integer such that $X(t)$ belongs to the set $\{X(s),\ldots,X(t-1)\}$ and the convention $0^{-1} = 0$ leads to $\beta(s-k,t) = \beta(s,t+k) = 0$ for $k \geq 1$.

2.2 Periodically Correlated Processes Class

A nonstationary process $X(\cdot)$ is called PC of period T [5] when its ACF is periodic of same period:

$$R(t+T,s+T) = R(t,s) \text{ for all } (t,s) \in \mathbb{Z}^2.$$

The relation between this processes class and the multivariate stationary processes one's is given in the following way: let define the jth component of the T-multivariate process $Y(\cdot) = \{Y(t), t \in \mathbb{Z}\}$ by,

$$Y_j(t) = X(j + T(t-1)) \text{ for } j = 1,\ldots,T, \; t \in \mathbb{Z}.$$

Then $Y(\cdot)$ is wide-sense stationary if and only if ([5]: Theorem 1) the associated scalar process $X(\cdot)$ is PC of period T.

According to the kind of relationship between the $R(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ functions, it is easy to see that $X(\cdot)$ is PC of period T if and only if $\beta(\cdot,\cdot)$ satisfies,

$$\beta(t+T,s+T) = \beta(t,s) \text{ for all } (t,s) \in \mathbb{Z}^2.$$

The periodicity property added to $\beta(\cdot,\cdot)$ other properties involves that the second order structure of these processes can be parametrized by T functions defined on $\mathbb{N}$, $\beta_t(k) = \beta(t,t-k)$, $t = 1,\ldots,T$ which are subject to only the following conditions

$$\beta_t(0) \geq 0 \text{ and } |\beta_t(k)| \leq 1, \; k \geq 1,$$

$$\beta_t(0) = 0 \Rightarrow \beta_t(k) = \beta_{(t+k)\bmod T}(k) = 0, \; k \geq 1,$$

$$|\beta_t(j)| = 1, \; j > 0 \Rightarrow \beta_t(k) = \beta_{(t+k-j)\bmod T}(k) = 0, \; k > j,$$

where $k\bmod T$ is the integer j in $[1,\ldots,T]$ such that $k = nT + j$, $n \in \mathbb{Z}$. We note D^T_β the satisfying above conditions functions set. Any function belonging to D^T_β is the $PACF$ of PC process of period T and from a such function, one can determinate the correspondent function $R(\cdot,\cdot)$ through the Levinson-Type algorithm described in the subsection below.

2.3 Levinson-Type Algorithm

According to the expressions of $\sigma^2(\cdot,\cdot)$ and $\sigma^{*^2}(\cdot,\cdot)$ in term of $\beta(\cdot,\cdot)$ (see [3]), we have for a PC process of period T,

$$\begin{aligned} \sigma^2(t;t-n) &= \sigma^2(k;k-n) = \sigma_k^2(n), \\ \sigma^{*^2}(t-n;t) &= \sigma^{*^2}(k-n;k) = \sigma_k^{*^2}(n), \end{aligned} \qquad k = t\bmod T.$$

Furthermore the coefficients in

$$\begin{aligned} \varepsilon(t;t-n) &= \sum_{j=0}^{n} a_k(n,j)X(t-j), & a_k(n,0) &= 1 \\ \varepsilon^*(t-n;t) &= \sum_{j=0}^{n} a_k^*(n,j)X(t-n+j), & a_k^*(n,0) &= 1, \end{aligned}$$

not unique in the degenerate case, depend only of $k = t\bmod T$. The Levinson-Type algorithm computes these coefficients (chosen by the procedure itself in the degenerate case) in order to establish the relationship between $\beta_k(n)$ and $R(k-j,k-n)$, $j = 0,\ldots,n-1$, through the quantity $\langle \varepsilon(k;k-n+1), X(k-n)\rangle$. The algorithm below is the Pham [7] one's adapted to provide the correspondence between $R(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ on the domain $\{1,\ldots,M\}^2$. We note r the integer such that $r = M\bmod T$.

Algorithm 2.1
For $k = 1,\ldots,T$*:*

$$R(k,k) = \beta_k(0) = \sigma_k^2(0) = \sigma_k^{*^2}(0).$$

For $n = 1,\ldots,M-1$, *with* $\sum_{j=1}^{0}\ldots = 0$ *and* $0^{-1} = 0$*:*

if $n \leq M-T$ *then* $A_1 = 1$, $A_2 = r$ *and* $A_3 = r+1$, *else*
if $n \geq M-T+1$ *and* $n \leq M-r$ *then* $A_1 = 1$, $A_2 = r$ *and* $A_3 = n\bmod T + 1$,
if $n \geq M-r+1$ *then* $A_1 = n\bmod T + 1$, $A_2 = r$ *and* $A_3 = T+1$,
for $k = A_1$ *to* A_2 *and* A_3 *to* T*:*

$$\beta_k(n) = \frac{R(k,k-n) + \sum_{j=1}^{n-1} a_k(n-1,j)R(k-j,k-n)}{\sigma_{k-1}^*(n-1)\sigma_k(n-1)},$$

$$\sigma_k^2(n) = \left[1 - |\beta_k(n)|^2\right]\sigma_k^2(n-1),$$

$$\sigma_k^{*^2}(n) = \left[1 - |\beta_k(n)|^2\right]\sigma_{k-1}^{*^2}(n-1),$$

$$a_k(n,n) = -\beta_k(n)\frac{\sigma_k(n-1)}{\sigma_{k-1}^*(n-1)},$$

$$a_k^*(n,n) = -\overline{\beta_k(n)}\frac{\sigma_{k-1}^*(n-1)}{\sigma_k(n-1)},$$

for $j = 1,\ldots,n-1$*:*

$$a_k(n,j) = a_k(n-1,j) + a_k(n,n)a_{k-1}^*(n-1,n-j),$$
$$a_k^*(n,j) = a_{k-1}^*(n-1,j) + a_k^*(n,n)a_k(n-1,n-j),$$

where the subscript $k-1 = 0$ *is replaced by* T*.*

3 EXTENSION OF AUTOCOVARIANCE COEFFICIENTS SEQUENCE

Before consider the problem of extension, let review some results of Pagano [11] needed for later discussions.

3.1 Periodic Autoregressive Processes

The autoregressive processes analogous for the PC processes is given by: a process $X(\cdot)$ is said to be Periodic Autoregressive of period T and order $(p_1,\ldots,p_T)$ if for all integer t,

$$\sum_{j=0}^{p_t} a_t(j)X(t-j) = \varepsilon(t), \quad a_t(0) = 1, \tag{1}$$

where $\{\varepsilon(t), t \in \mathbb{Z}\}$ is a sequence of zero-mean uncorrelated variables with $Var\{\varepsilon(t)\} = \sigma_t^2$, $p_t = p_{t+T}$, $\sigma_t^2 = \sigma_{t+T}^2$ and $a_t(j) = a_{t+T}(j)$, $j = 1,\ldots,p_t$.

The correspondence with the autoregressive multivariate stationary processes is stated in [11] by the following theorem.

Theorem 3.1 *(Pagano [11]) Let $Y(\cdot)$ be the T-multivariate stationary process associated with $X(\cdot)$. Then $Y(\cdot)$ admits* *the representation,*

$$\sum_{j=0}^{p} A(j)Y(t-j) = e(t), \quad A(0) = I_T, \tag{2}$$

where $\{e(t), t \in \mathbb{Z}\}$ is a sequence of zero-mean uncorrelated T-dimensional variables such that $\Gamma = Var\{e(t)\}$ is positive definite matrix, if and only if $X(\cdot)$ is PAR of period T and order $(p_1, \ldots, p_T)$ with positive $\sigma_1^2, \ldots, \sigma_T^2$ and $p = \max_j [(p_j - j)/T] + 1$, where, for any real x, $[x] = j$ is the integer part of x.

Furthermore, the relationships between parameters of both models are given by,

$$A(j) = L^{-1}A'(j), \ j = 1, \ldots, p, \quad \Gamma = L^{-1}D\overline{L^{-1}}^T,$$

where $\bar{\cdot}^T$ denotes the conjugate transpose and $A(\cdot)$, L and D are $T \times T$ matrices determined by,

$$\begin{array}{ll} L_{kj} = a_k(k-j) \text{ for } j \leq k, \text{ else } 0, & \\ A'_{kj}(v) = a_k(Tv + k - j), \ v = 1, \ldots, p & \end{array} \quad i, j = 1, \ldots, T.$$

$$D = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_T^2).$$

We can easily show (see proof and comment of [10]: Theorem 1) that the Theorem 3.1 is available even in the degenerate case (i.e. Γ is singular). The coefficients of the L and $A'(\cdot)$ matrices are not uniquely defined and some elements on the D diagonal vanish.

We can assume without loss of generality that $e(\cdot)$ in (2) and $\varepsilon(\cdot)$ in (1) are the innovation processes of $Y(\cdot)$ and $X(\cdot)$. Replacing the innovation process decompositions by the forward partial innovation one's in (2) and (1) (where $p = n$ and $p_j = nT - 1 + j$, $j = 1, \ldots, T$, $n \in \mathbb{N}$), we obtain in the same Pagano's way the following result useful later,

$$\Gamma_n = L_n^{-1} D_n \overline{L_n^{-1}}^T, \quad n \in \mathbb{N}, \tag{3}$$

where Γ_n is the n-*th* order forward partial innovation of $Y(\cdot)$ variance,

$$D_n = \mathrm{diag}(\sigma_1^2(nT), \sigma_2^2(nT+1), \ldots, \sigma_T^2((n+1)T-1)),$$

and L_n is a lower triangular matrix (with 1 on the diagonal) defined by, for $k, j = 1, \ldots, T$:

$$\{L_n\}_{kj} = a_k(nT - 1 + k, k - j) \text{ if } k \leq j, \text{ else } 0.$$

3.2 The Extension Problem

Let $R(t, s)$, $t, s = 1, \ldots, M$, be the T-periodic function values. According to the previous section, we can treat the extension problem, stated in the introduction, in terms of partial autocorrelation. Starting from the data $R(t, s)$, the Algorithm 2.1 yields for $k = 1, \ldots, T$, $\beta_k(n)$, $0 \leq n \leq mo(k)$, where $mo(k) = M - r - 1 + k$ if $k \leq r$, else $M - T - r - 1 + k$. The values $R(t, s)$ represent ACF one's if and only if the coefficients $\beta_k(n)$ satisfy the D_β^T constraints. The whole extensions of such functions are so described through the extensions of $\beta_k(n)$ remaining in D_β^T. Notice that the degenerate case arises from the finite order singularities on $\{1, \ldots, M\}$. What is translated by the coefficients $\beta_k(n)$ of magnetude equal to 1.

3.3 The Maximum Entropy Method

For the scalar stationary processes, it is well known that the MEM solution is the one for which the innovation process variance is maximum. The variance σ_t^2 of the innovation process for a PC process is T-periodic and given by [4],

$$\sigma_t^2 = \beta_k(0) \prod_{n=0}^{+\infty} \left[1 - |\beta_k(n)|^2\right], \quad k = t \bmod T.$$

Several of these T variances can vanish in the degenerate case. To extend the MEM to this process class, we suggest so the following approach which will be justify later. This method consists in choosing among the whole extensions the one making maximum the quantity,

$$\prod_{k=1, k \notin S}^{T} \sigma_k^2, \quad \text{where } S = \left\{k, \sigma_k^2(mo(k)) = 0\right\}.$$

According to the σ_k^2 expression, this is equivalent to maximize separately each terms of the above product and put $\beta_k(n) = 0$, for $k = 1, \ldots, T$, $n > mo(k)$. Thus [3] the solution is PAR of period T and order $(mo(1), \ldots, mo(T))$ and the parameters of this model are provided by the Algorithm 2.1 in the following way:

$$\begin{array}{l} \sigma_k^2 = \sigma_k^2(mo(k)), \\ a_k(j) = a_k(mo(k), j), \ j = 1, \ldots, mo(k), \end{array} \quad k = 1, \ldots, T.$$

In order to justify this method, we consider the associated extension problem in the multivariate stationary case. Let $X(\cdot)$ be a process which ACF coincides with the data on the domain $\{1, \ldots, T\}^2$ and $Y(\cdot)$ be the T-multivariate process associated with it. If we note the autocorrelation matrix of $Y(\cdot)$, $R_k = E\left\{Y(t+k)\overline{Y(t)}^T\right\}$, we have $\{R_k\}_{ij} = R(i + kT, j)$, $i, j = 1, \ldots, T$. When $M = (n+1)T$, $n \geq 0$, the problem considered coincides with the extension of $R_0, \ldots, R_n$ one's.

The MEM, proposed by Burg [9] in the nondegenerate case, consists in finding the multichannel spectrum $P(\cdot)$ that satisfies

$$\int_{-\pi}^{+\pi} P(\omega) e^{i\omega k} d\omega = R_n, \quad k = -n, \ldots, n,$$

and maximizes

$$\int_{-\pi}^{+\pi} \ln \det P(\omega) d\omega.$$

On the other hand, we have ([13]: Theorem 7.10),

$$\int_{-\pi}^{+\pi} \ln \det P(\omega) d\omega = \ln \det \Gamma_\infty,$$

where $\Gamma_\infty = \lim_{n \to +\infty} \Gamma_n$ represents the covariance matrix of the innovation processes. The Γ_n decomposition (3) gives

$$\det \Gamma_\infty = \lim_{n \to +\infty} \det D_n = \prod_{k=1}^{T} \sigma_k^2.$$

This shows the both methods coincides for this situation.

In [10], Inouye proposes to extend the MEM to the particular degenerate case where exists a purely nondeterministic T-dimensional process associated with a solution of $R_0, \ldots, R_n$ extension problem. The solution maximizing the entropy is the one for which the covariance matrix of the innovation process Γ_∞ satisfies

$$\Gamma_\infty \geq \tilde{\Gamma}_\infty, \tag{4}$$

where $\Gamma_\infty \geq \tilde{\Gamma}_\infty$ means $\Gamma_\infty - \tilde{\Gamma}_\infty$ nonnegative definite and $\tilde{\Gamma}_\infty$ is the covariance matrix of the innovation process associated with any other solution. In fact, this method consists in fitting an AR model to the data $R_0, \ldots, R_n$. The decomposition (3) involves that the solution we propose is such that $\Gamma_\infty = \Gamma_n$ and consequentely is identical with the Inouye's one. Let us remark that our method is equivalent to maximize Γ_∞ (in the sense of (4)) even if M is not a multiple of T.

3.4 Cyclo-Correlation Functions

The second order structure of a PC process can be parametrized by its T cyclo-correlation functions $B_1(\cdot), \ldots, B_{T-1}(\cdot)$ defined by [5],

$$R(t+n,t) = \sum_{k=0}^{T-1} B_k(n) e^{\frac{2\pi i k t}{T}}, \quad (t,n) \in \mathbb{Z}^2.$$

For convenience the definition of functions $B_k(n)$, $k = 0, \ldots, T-1$, is completed for integers k with the equality $B_k(n) = B_{k+T}(n)$. Alpay *et al.* [12] have considered the extension of such functions from the data $B_k(n)$, $k = 0, \ldots, T-1$, $0 \le n \le N$, in the nondegenerate case. These data represent the first values taken by cyclo-correlation functions if and only if the set of matrices $\{B_{lj}(\cdot)\}_{l,j=0,\ldots,T-1}$ defined by,

$$B_{lj}(n) = B_{j-l}(n) e^{\frac{2\pi i l n}{T}}, \quad n = 0, \ldots, N, \tag{5}$$

is the one of some T-dimensional stationary process autocorrelation matrices. Then the whole extension of the data $B_k(n)$ is described through the $B(n)$ one's respecting (5). Thus this procedure consists in extending a particular sequence of autocorrelation matrices.

This problem is equivalent to the one of $R(\cdot,\cdot)$ extension from the data $R(t, t-n)$, $t = 1, \ldots, T$, $0 \le n \le N$. The Algorithm 2.1 can be adapted for this domain and the $PACF$ allows to resolve this one even in the degenerate case. Notice that one can use another way by considering the PC process $PACF$ associated with the second order structure defined by the $T \times T$ matrices $B(\cdot)$.

4 CONCLUSION

We have shown that the $PACF$ is a good parametrization for the second order structure of PC processes. This allows us to resolve the extension of autocovariance coefficients sequence problem in an elegant way even in the degenerate case. Then we have extended the MEM to the degenerate case and shown that the solution is given by a PAR process.

REFERENCES

[1] G. U. Yule, "On the Theory of Correlation for any Number of Variables, treated by a New System of Notation", Roy. Soc. Proc., vol. 79A, pp. 85-96, 1907.

[2] F. L. Ramsey, "Characterization of the Partial Autocorrelation Function", Ann. Stat., n.2, pp. 1296-1301, 1974.

[3] S. Dégerine and S. Lambert, "Evolutive Instantaneous Spectrum Associated with Partial Autocorrelation Function for Nonstationary Time Series", to be appear in Proc. TFTS-96, june 18-21, Paris, France, 1996.

[4] S. Dégerine et S. Lambert, "Fonction d'autocorrélation partielle des processus à temps discret non stationnaires et spectre évolutif associé", Rap. Tech., LMC (URA 397), RT 156, 1996.

[5] E. G. Gladysev, "Periodically Random Sequences", Soviet Mat., vol. 2, pp. 385-388, 1961.

[6] H. Sakai, "Circular Lattice Filtering using Pagano's Method", IEEE Trans. ASSP, vol. ASSP-30, pp. 279-287, 1982.

[7] D. T. Pham, "Fast Scalar Algorithms for Multivariate Partial Correlations and their Empirical Counterparts", Rapport de Recherche, LMC (URA 397), RR 903-M-, 1992.

[8] S. Dégerine, "Canonical Partial Autocorrelation Function of a Multivariate Time Series", Ann. Stat. 18, n.2, pp. 961-971, 1990.

[9] J. P. Burg, "Maximum Entropy Spectral Analysis", Ph. D. dissertation, Dep. Geop. Standford Uni., CA, 1975.

[10] Y. Inouye, "Maximum Entropy Spectral Estimation for Regular Time Series of Degenerate Rank", IEEE Trans. ASSP, vol. ASSP-32, pp. 733-740, 1984.

[11] M. Pagano, "On Periodic and Multiple Autoregressions", Ann. Stat. 6, n.6, pp. 1310-1317, 1978.

[12] D. Alpay, A. Chevreuil et P. Loubaton, "Extension des cyclocorrélations et estimation spectrale au sens du maximum d'entropy pour les processus aléatoires périodiquement corrélés", 15 ème Colloque GRETSI-Juan-Les-Pins, pp. 33-36, 1995.

[13] N. Wiener and P. Masani, "The Prediction Theory of Multivariate Stochastic Processes, I", Acta. Math., vol. 98, pp. 111-150, 1957.