Symmetric Delay Factorization: A Generalized Theory for ParaUnitary Filter Banks

Patrick Rault and Christine Guillemot
CCETT
dpt RCS/ATI
4, rue Clos Courtel. 35 512 Cesson-Sévigné, France. Tel: +33 99 12 44 35
`E-mail: prault@ccett.fr`

ABSTRACT

The Symmetric Delay Factorization (SDF) is introduced in [1] for synthesizing linear phase paraunitary filter banks and is applied successfully in [2] for designing Time Varying Filter Banks (TVFB). This paper describes a minimal and complete generalized symmetric delay factorization theory valid for a larger class of paraunitary filter banks. The approach presented here provides a unifying framework for linear phase paraunitary filter banks including linear phase Lapped Orthogonal Transforms (LOT) and for cosine-modulated filter banks, this for an arbitrary number of channels (odd or even). This generalized theory opens new perspectives in the design of time varying filter banks used for image and video compression, especially in the framework of region or object based coding. The generalized symmetric delay factorization relying on lattice structure representations leads also naturally to fast implementation algorithms.

1 INTRODUCTION

Paraunitary filter banks can be synthesized by decomposing the filter bank analysis and synthesis matrices into their polyphase components resulting in the so-called Polyphase Transfer Matrices (PTM). Design techniques relying on PTM structures are more efficient than the approaches based on a direct decomposition of the filter bank [3]. In particular, the number of parameters to be optimized is significantly reduced and this is essential for example in the design of time varying filter banks for which the number of parameters grows rapidly [2]. The PTM leads also directly to a fast implementation algorithm of the filter bank.

The PTM, being paraunitary, can be decomposed into a series of orthogonal matrices and delay stages [1]. This decomposition is called the Symmetric Delay Factorization (SDF). The symmetric delay factorization provides a lattice structure decomposition of the polyphase transfer matrix of the filter bank [3]. A technique of design of time varying filter banks [2, 4, 5, 6] based on a symmetric delay factorization of paraunitary filter banks is developed in [2]. In comparison with other structures of decomposition of the PTM [3], the SDF presents the advantage of decomposing the polyphase transfer matrix into only square orthonormal matrices, even at the boundary of finite length signals where TVFB are being used. This simplifies significantly the design procedure of TVFB.

However, the symmetric delay factorization technique introduced so far in the literature [1] applies only to linear phase paraunitary filter banks. In addition, the approaches proposed so far lead to non square matrices at the boundary of finite length signals when the number of channels is odd [1]. For other classes of paraunitary filter banks such as the cosine-modulated filter banks now widely used in speech, audio and image coding, no structure relying on a symmetric delay factorization of the PTM has been provided so far. The first key result of the paper is the development of a new minimal and complete symmetric delay factorization of the polyphase transfer matrix valid for a large class of paraunitary filter banks including cosine-modulated filter banks. A second key issue is that the approach does not make any assumption on the parity of the number of channels and leads also for the case where the number of channels is odd to only square orthonormal matrices, even at the boundary of finite length signals. Some preliminary results, introducing a "Pseudo-Symmetric" Delay Factorization, for the case where the number of channels is odd have been presented in [7]. The approach finds very strong interest in the design of time varying filter banks.

2 GENERALIZED SYMMETRIC DELAY FACTORIZATION

Let us consider a M-channel filter bank, composed of Finite Impulse Response filters of length $L = M(N+1)$, where N is an arbitrary integer. Let also

$$(m_l, m_u) = \begin{cases} (\frac{M}{2}, \frac{M}{2}) & \text{if } M \text{ is even,} \\ (\frac{M-1}{2}, \frac{M+1}{2}) & \text{if } M \text{ is odd.} \end{cases} \quad (1)$$

The following theorem, providing a cascade form structure of the Polyphase Transfer Matrix of a paraunitary filter bank, is proved in the case where the paraunitary system is linear phase and also in the case where the system is a cosine-modulated filter bank. This cascade form structure represents the Symmetric Delay Factorization of the filter bank.

theorem 1 *Let $E^N(z)$ be a FIR paraunitary matrix of polynoms of maximum degree N, representing the polyphase transfer matrix of a M channel filter bank. The matrix $E^N(z)$ can be expressed as $E^N(z) = \sum_{n=0}^{N} z^{-n} P_n^N$, where $P_N(n)$ are $M \times M$ matrices [3, 2]. If*

$$rank(P_N^N) \leq M/2 \; and \; rank(P_0^N) \leq M/2, \quad (2)$$

then the matrix $E^N(z)$ can be decomposed as the product

$$E^N(z) = V_N(z)V_{N-1}(z)...V_1(z)V_0, \quad (3)$$

where, the matrices $V_n(z)$, for $n > 0$, are given by

$$\begin{aligned} V_{2i+1}(z) &= B_{2i+1}\Lambda_o(z) \\ B_{2i+1} &= \begin{pmatrix} B_{2i+1,1} & B_{2i+1,2} \end{pmatrix} \\ \Lambda_o(z) &= \begin{pmatrix} I_{m_u} & O \\ O & z^{-1}I_{m_l} \end{pmatrix}, \end{aligned} \quad (4)$$

for n odd and by,

$$\begin{aligned} V_{2i}(z) &= B_{2i}\Lambda_e(z) \\ B_{2i} &= \begin{pmatrix} B_{2i,1} & B_{2i,2} \end{pmatrix} \\ \Lambda_e(z) &= \begin{pmatrix} I_{m_l} & O \\ O & z^{-1}I_{m_u} \end{pmatrix}, \end{aligned} \quad (5)$$

for n even. $B_{2i+1,1}$ and $B_{2i+1,2}$ are matrices of respective size $M \times m_u$ and $M \times m_l$, and $B_{2i,1}$ and $B_{2i,2}$ are matrices of respective size $M \times m_l$ and $M \times m_u$ such that the matrices B_{2i} and B_{2i+1} are unitary. The matrices $B_{2i+1,1}$ and $B_{2i+1,2}$ must also be orthogonal respectively to the matrices $B_{2i,1}$ and $B_{2i,2}$.

Note that in practice, the matrices B_i as explained in section (5) are Givens plane rotations. Note also that in the case where the number of channels M is odd, m_u being different from m_l, a "Pseudo-Symmetric" Delay Factorization is obtained.

3 PRELIMINARY RESULTS

In order to prove theorem 1, let us first demonstrate the two following preliminary results.

theorem 2 *Let $E^N(z)$ be a FIR paraunitary matrix of polynoms of maximum degree N, representing the Polyphase Transfer Matrix of a M channel filter bank. The matrix $E^N(z)$ verify $E^N(z) = \sum_{n=0}^{N} z^{-n} P_n^N$, where P_n^N are $M \times M$ matrices. If the filter bank is linear phase, then the matrices P_0^N and P_N^N verify*

$$rank(P_n^N) \leq \frac{M}{2} \; \forall n \in \{0, N\}. \quad (6)$$

Proof: Let $h_k(z)$ be the k^{th} filter of length $L = M(N+1)$ of a M-channel filter bank of Polyphase Transfer Matrix $E^N(z)$. The matrix $E^N(z)$ has elements in the field of polynomials of z. The elements of the matrix $E^N(z)$, $[E^N(z)]_{k,l}$, are the z-transforms of the sequences [2]

$$e_N(n)_{k,l} = h_k(nM + M - 1 - l), \quad (7)$$

and $[P_n^N]_{k,l} = [e^N(n)]_{k,l}$. In [1], it is demonstrated that a linear phase paraunitary filter bank has m_u symmetric filters and m_l antisymmetric filters.
The impulse response of the k^{th} Filter verifies $h_k(n) = sgn(k)h_k(L-1-n)$, where $sgn(k) = 1$ if h_k is symmetric, and $sgn(k) = -1$ if h_k is antisymmetric. As a consequence, the elements $[e^N(n)]_{k,l}$ verify

$$[e^N(n)]_{k,l} = sgn(k)[e^N(N-n)]_{k,M-1-l}. \quad (8)$$

So, the k^{th} row of P_N^N is the time-reverse of the k^{th} row of $sgn(k)P_0^N$, and

$$rank(P_n^N) = rank(P_{N-n}^N). \quad (9)$$

Since the filter bank is paraunitary, the matrices P_0^N and P_N^N verify $P_0^N(P_N^N)^T = (P_0^N)^T P_N^N = 0$, and span two orthogonal spaces of maximum dimension M. Since P_0^N and P_N^N span two orthogonal spaces of the same rank, obviously they verify $rank(P_0^N) = rank(P_N^N) \leq \frac{M}{2}$.

theorem 3 *Let $E^N(z)$, be a FIR paraunitary matrix of polynoms of maximum degree N, representing the Polyphase Transfer Matrix of a M channel filter bank. The PTM $E^N(z)$ verifies $E^N(z) = \sum_{n=0}^{N} z^{-n} P_n^N$, where P_n^N are $M \times M$ matrices. If the filter bank is a cosine-modulated filter bank, then the matrices P_n^N, $n = 0, ..., N$ verify,*

$$rank(P_n^N) \leq \begin{cases} m_u & \text{if } n \text{ is even} \\ m_l & \text{if } n \text{ is odd.} \end{cases} \quad (10)$$

Proof: Let us consider a paraunitary cosine-modulated filter bank obtained by modulation of a linear phase lowpass prototype $h(n)$ of length $L = (N+1)M = 2pM$. The impulse response of the k^{th} filter can be expressed as [8, 9, 10]

$$\begin{aligned} h_k(n) &= c_{k,n} h(n) \quad (11)\\ \text{where, } c_{k,n} &= cos\Big((2k+1)\big((n - \frac{L-1}{2})\frac{\pi}{2M} - \frac{\pi}{4}\big)\Big), \end{aligned}$$

Let C_e and C_o the $M \times M$ square matrices defined by

$$C_e = \{c_{k,n}\}_{0 \le k \le M-1, 0 \le n \le M-1}, \quad (12)$$
$$C_o = \{c_{k,n}\}_{0 \le k \le M-1, M \le n \le 2M-1}. \quad (13)$$

The equations

$$\begin{aligned} c_{k,n} &= c_{k,n+2M}, \quad (14)\\ c_{k,pM+n} &= c_{k,(p+1)M-n}, \ \forall p \in \{0, \dots, N\}\\ c_{k,(2p+1)M+m_l} &= 0, \text{ if } M \text{ is odd} \end{aligned}$$

expressing, in the modulation function $c_{k,n}$, m_l symmetric relations for $n \in \{0, \dots, M-1\}$, and m_l symmetric relations for $n \in \{M, \dots, 2M-1\}$ can be verified easily. The ranks of the matrices C_e and C_o are thus $rank(C_e) = m_u$ and $rank(C_o) = m_l$. Since the matrices P_{2i}^N and P_{2i+1}^N can be obtained from C_e and C_o by the relations [2, 11]

$$\begin{aligned} [P_n^N]_{k,l} &= c_{k,L-1-nM-l} h(L-1-nM-l) \quad (15)\\ P_{2i}^N &= -C_o g^{2i} \quad (16)\\ P_{2i+1}^N &= C_e g^{2i+1} \quad (17) \end{aligned}$$

where g^n is a vector of length M defined by $g^n(l) = h(L-1-nM-l)$, then $rank(P_0^N) \le m_u$ and $rank(P_1^N) \le m_l$.
remark: In practice, in the case where M is odd, in order to obtain a paraunitary filter bank, the prototype filter must be such that $rank(P_N^N) \le m_l$ [9]. As for the case where M is odd, $h(m_u) = 0$ [11], the paraunitary cosine modulated filter banks turn out to be such that,

$$rank(P_n^N) \le M/2 \ \forall n \in \{0, N\}. \quad (18)$$

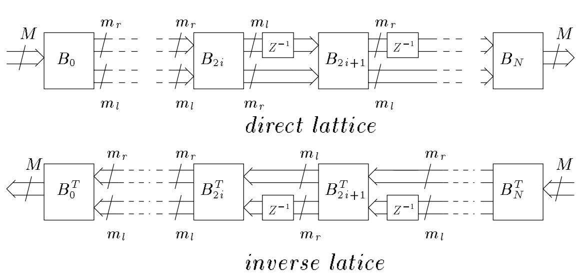

Figure 1: *M channel lattice*

4 GENERALIZED SYMMETRIC DELAY FACTORIZATION: PROOF OF THEOREM 1

Theorem 1 can be proved by using a degree reduction procedure as outlined below. Let us consider $E^N(z)$, PTM matrix of degree N of the M-channel filter bank. The matrix $E^N(z)$ verifies $E^N(z) = \sum_{n=0}^N z^{-n} P_n^N$. Let us assume that the matrix $E^N(z)$ is such that the theorem 1 assumptions (2) are verified. As shown in theorems 2 and 3, this is always verified by linear phase paraunitary filter bank and by cosine-modulated filter banks. Let us introduce the orthogonal matrix $B_N = (B_{N,1}, B_{N,2})$, such as,

$$B_{N,1}^T P_N^N = 0 \quad (19)$$
$$B_{N,2}^T P_0^N = 0 \quad (20)$$
$$B_{N,1} B_{N,2}^T = B_{N,1}^T B_{N,2} = 0 \quad (21)$$

The existence and the paraunitarity of the two matrices $B_{N,1}$ and $B_{N,2}$ can be proved from theorems 2 or 3. The proof is provided in [11]. Let the matrix $E^{N-1}(z)$ be

$$E^{N-1}(z) = V_N(z^{-1})^T E^N(z) = (B_N \Lambda_N(z^{+1}))^T E^N(z) \quad (22)$$

where $\Lambda_N(z) = \Lambda_e(z)$ if N is even, or $\Lambda_N(z) = \Lambda_o(z)$ if N is odd. $E^{N-1}(z)$ can be rewritten $E^{N-1}(z) = (B_N \Lambda_N(z^{+1}))^T (\sum_{n=0}^N z^{-n} P_n^N)$. From equations (20,4,5), the noncausal term of $E^{N-1}(z)$ is cancelled, and from equation (19), the degree of $E^{N-1}(z)$ is lower or equal to $N-1$.
$E^{N-1}(z)$ cannot have degree smaller than $N-1$ (since $E^N(z)$ has degree N and V_N has degree 1). So the degree of $E^{N-1}(z)$ is precisely $N-1$, and the factorization is minimal. Since the unitary transformation $V_N = B_N \Lambda_N(z^{+1})$ is a bijection, the matrix $E^{N-1}(z)$ is also paraunitary and we can write

$$E^N(z) = (B_N \Lambda_N(z^{-1})) E^{N-1}(z) \quad (23)$$

In addition, the bijection property of V_N guarantees that

$$rank(P_0^{N-1}) \le rank(P_0^N), \quad (24)$$
$$rank(P_{N-1}^{N-1}) \le rank(P_N^N). \quad (25)$$

Hence $E^{N-1}(z)$ verifies the theorem 1 assumptions (2. Repeating the degree reduction procedure N times, we obtain the complete decomposition (3) where V_0 is a nonzero vector. Then the filter can be implemented the filter by the lattice structure of figure 1. In the 2-channel case, we obtain the classical lattice structure, similarly in [3], and valid for all 2-channel paraunitary filer banks.

5 APPLICATION TO THE DESIGN OF TIME VARYING FILTER BANKS

The lattice structure presents well known advantages:
-It provides fast implementation algorithms,
-The orthogonality (paraunitarity) is inherent to the structure,
-The perfect reconstruction property is preserved even in presence of quantization of the lattice coefficients,
-Any change of the plane Givens rotations (each block of the lattice) allows to span all perfect reconstruction solutions described by the structure.

The symmetric delay factorization turns out to be a very powerful tool for the design of Time Varying filter banks useful for the processing of time bounded signals or in the transition phase when switching from one filter bank to another one to process two regions with different characteristics. From the decomposition provided by theorem 1, the bounded lattice structure contains only orthogonal square blocks (Givens plane rotations) as shown in figure 3. The blocks in grey are transient blocks (analogue to the transient filters necessary in time varying filtering [5]) which are orthogonal, so the bounded filter bank verifies the perfect reconstruction property.

This structure allows to proccess a signal of arbitrary length of the form $kM + r + l$ with a cosine-modulated filter bank or a linear phase paraunitary filter with an arbitrary number of channels. Each block of the lattice structure can be represented and synthesized by using Givens rotations that inherently guarantee the orthogonality and the perfect reconstruction property. The parameters α_i of the Givens rotations B_i can be optimized so that the filters $h_k(n)$ derived from the matrices B_i by equations (3-5) verify some criteria such as the classical frequency selectivity or the maximum coding gain.

6 CONCLUSION

A new minimal and complete symmetric delay factorization formalism valid for a large class of paraunitary filter banks including linear phase filter banks, cosine-modulated filter banks, all the 2-channel cases..., has been described without assumption on the parity of the number of channels. It is shown that this formalism opens new perspectives in the design of paraunitary filter banks and in particular in the design of time varying filter banks. The approach allows to use Givens rotations in the design procedure that inherently guarantee the perfect reconstruction. This approach, by preserving the square shape of the different blocks of the matrices of transition to be designed at the border of a finite length signal, provides a simplified method for synthesizing and implementing time-varying filter banks.

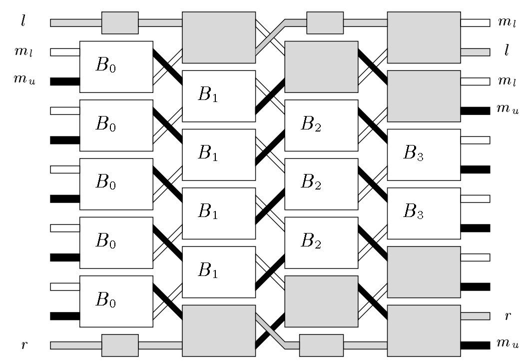

Figure 2: *Temporal bounded lattice*

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