SIMPLIFIED DESIGN OF LINEAR-PHASE PROTOTYPE FILTERS FOR MODULATED FILTER BANKS

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ABSTRACT

In this paper a simplified design of linear-phase prototype filters for almost perfect reconstruction modulated filter banks will be presented. It is based on an improved frequency-sampling design where the frequency response of an easily designable Nyquist filter is shaped such that the prototype constraints will be approximately satisfied. This method does not involve any coefficient optimization and results in a computationally more efficient, faster and more stable design process, which is especially well suited for longer filters.

1 INTRODUCTION

Several methods for designing linear-phase prototype-filters for almost perfect reconstruction modulated filter banks have been developed up to now.

In [1, 2, 3] a nonlinear error function in the frequency domain is constructed, which is then minimized using an unconstrained optimization algorithm. Recently, Nguyen proposed an improved design method [4], where additional time domain constraints are introduced into the design process. However, in nonlinear optimization, the objective function to be minimized exhibits many local minima. Thus the properties of the resulting filter highly depend on the quality and also on the availability of an optimization algorithm. Another disadvantage is the large computational cost being connected with these methods.

Design methods, which are avoiding any nonlinear optimization, suffer from other drawbacks. Closed form designs like [5] are easy to compute, but they often lack design flexibility, since design parameters like transition bandwidth and stopband / passband weighting cannot be chosen independently. The approach described in [6] is based on a Remez design and yields equiripple filters with good stopband attenuation, but does not allow to specify the transition bandwidth for the design. Like the method in [7], which is based on frequency sampling, it is only restricted to wider transition bands.

In this paper, a simple method for designing linear-phase approximate square-root Nyquist prototype FIR filters for M-band almost-perfect reconstruction modulated filter banks is proposed, which is related to [7] but is based on an improved frequency-sampling design.

2 PROTOTYPE FILTER SPECIFICATION

The frequency response $H(e^{j\Omega})$ of an ideal real-valued square-root Nyquist lowpass prototype $h(n)$ satisfies the following conditions, where $K = 2M$ for cosine-modulated filter banks with M real subbands [1, 2, 3, 8] and $K = M$ in the complex modulated M-band case [5]:

$$\left|H(e^{j\Omega})\right|^2 + \left|H(e^{j(\Omega-2\pi/K)})\right|^2 = \begin{cases} 1 & 0 \le \Omega < \frac{2\pi}{K}, \\ 0 & \Omega < -\frac{2\pi}{K},\ \Omega \ge \frac{4\pi}{K}, \\ \text{arbitrary} & \text{elsewhere}, \end{cases} \tag{1}$$

$$\left|H\left(e^{j\Omega}\right)\right| = \begin{cases} 0 & |\Omega| \ge \frac{2\pi}{K}, \\ \text{arbitrary} & \text{elsewhere}. \end{cases} \tag{2}$$

The exact power-complementarity between the prototype and its shifted version for $0 \le \Omega < 2\pi/K$ in (1) removes all amplitude distortion, and the infinitely high stopband attenuation of the lowpass prototype for $|\Omega| \ge 2\pi/K$ suppresses all aliasing components between non-adjacent subbands, where aliasing in adjacent subbands is cancelled out by the filter bank itself. However, FIR filters cannot satisfy the infinitely high stopband attenuation in (2) and linear-phase FIR filters additionally cannot exactly satisfy the power-complementarity constraint, so that additional distortion is introduced to the output of the filter bank.

3 DESIGN PREREQUISITES

For convenience we define $S = (N-1)/2$, where N denotes the filter length. If $h(n) = h(N-1-n)$ holds for an arbitrary linear-phase filter $h(n)$, then its frequency response is given by $H(e^{j\Omega}) = H_0(e^{j\Omega})e^{-jS\Omega}$, where $H_0(e^{j\Omega})$ denotes the real-valued symmetric amplitude response. We show now that $H_0(e^{j\Omega})$ can be expressed by cosine modulation of the filter coefficients $h(n)$, where we limit ourselves to odd-length filters. However, the solution for even-length filters is analogous.

The amplitude response of a linear-phase filter $h(n)$ can be written as

$$H_0(e^{j\Omega}) = \mathbf{h}^{\mathrm{T}}\mathbf{c}(\Omega) \tag{3}$$

with the length $(N+1)/2$ vectors

$$\mathbf{h} = [2h(0), 2h(1), \ldots, 2h(S-1), h(S)]^{\mathrm{T}} \tag{4}$$

$$\text{and} \quad \mathbf{c}(\Omega) = [\cos(S\Omega), \cos((S-1)\Omega), \ldots, 1]^{\mathrm{T}}.$$

Since $h(n)$ is linear-phase, the sampled amplitude response $H_{s0}(k) = H_0(e^{j\Omega_k})$, $\Omega_k = \frac{2\pi}{L}k$, $k = 0, \ldots, L-1$, shows the symmetry $H_{s0}(k) = H_{s0}(L-k)$, where L denotes the number of sampling points in the frequency domain. Thus, it is sufficient to consider only the first $L/2+1$ frequency samples. With

$$\tilde{\mathbf{h}} = [H_{s0}(0), H_{s0}(1), \ldots, H_{s0}(L/2)]^{\mathrm{T}}$$

and the DCT-I matrix

$$\mathbf{C} = [\mathbf{c}(0),\ \mathbf{c}(2\pi/L),\ \cdots,\ \mathbf{c}(\pi)]^{\mathrm{T}} \in \mathbb{R}^{(L/2+1)\times(S+1)}, \tag{5}$$

the sampled amplitude response is given in matrix notation by

$$\tilde{\mathbf{h}} = \mathbf{C}\mathbf{h}. \tag{6}$$

Similarly, it can be shown, that the inverse DFT is given by

$$\mathbf{h}_{\mathrm{I}} = \mathbf{C}^{\mathrm{T}}\tilde{\mathbf{h}}_{\mathrm{I}}, \tag{7}$$

where

$$\tilde{\mathbf{h}}_{\mathrm{I}} = \mathbf{D}\tilde{\mathbf{h}}, \quad [\mathbf{D}]_{ij} = \frac{1}{L}\begin{cases} \delta_{ij} & i=0, i=L/2, \\ 2\delta_{ij} & i=1,\ldots,L/2-1, \end{cases} \tag{8}$$

$j = 0, \ldots, L/2$; δ_{ij} denoting the Kronecker symbol and

$$\mathbf{h}_{\mathrm{I}} = [h(0), h(1), \ldots, h(S)]^{\mathrm{T}}. \tag{9}$$

4 DESIGN METHOD

The design procedure to be proposed here is based on a frequency-sampling design, where the *desired* frequency response is constructed such that it can be represented almost exactly by a linear-phase FIR filter [9]. This process can be summarized as follows:

1. Design of a for the present arbitrary linear-phase FIR filter $g(n)$ with frequency response $G(e^{j\Omega})$.

2. Modification of $G(e^{j\Omega})$ such that the constraint (1) is approximated in a least-squares sense.

3. Inverse discrete Fourier transform of the sampled modified frequency response yields the impulse response of the desired prototype filter $h(n)$.

4.1 Odd-length filters

Starting point is the design of a linear-phase K-th band or Nyquist(K) FIR filter $g(n)$ of odd length N and transition bandwidth b_g, where its amplitude response $G_0(e^{j\Omega})$ satisfies the flatness constraint

$$\sum_{k=0}^{K-1} G_0\left(e^{j(\Omega - k\cdot 2\pi/K)}\right) = 1. \tag{10}$$

The design of such a Nyquist(K) filter with transition bandwidth b_g $(0 < b_g < 2\pi/K)$ and filter length N can be carried out e.g. with an eigenfilter approach [8], which is used throughout this paper and has been chosen, since it offers more design flexibility. However, it is possible to use other design methods for $g(n)$ here.

The amplitude response $H_0(e^{j\Omega})$ of the *desired* zero-phase square-root Nyquist filter is now represented by

$$H_0\left(e^{j\Omega}\right) = \begin{cases} \sqrt{G_0(e^{j\Omega})} & \text{for} \quad 0 \le |\Omega| < \Omega_s, \\ G_0(e^{j\Omega}) & \text{for} \quad \Omega_s \le |\Omega| < \pi, \end{cases} \tag{11}$$

with $\Omega_s = 2\pi/K + b_g/2$, where Ω_s denotes the stopband-edge frequency. The DCT matrix $\mathbf{C}$ in (5) can be divided into two submatrices $\mathbf{C}_1$ and $\mathbf{C}_2$ according to

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix},$$

where $\mathbf{C}_1 \in \mathbb{R}^{L_1\times(S+1)}$, $L_1 = \lceil \Omega_s L/(2\pi) \rceil$, contains the lower and $\mathbf{C}_2 \in \mathbb{R}^{L_2\times(S+1)}$, $L_2 = L/2+1-L_1$, the upper frequency rows of $\mathbf{C}$, respectively. With (3) and (6) eq. (11) can now be written as

$$\tilde{\mathbf{h}} = \begin{bmatrix} \sqrt{\mathbf{C}_1\mathbf{g}} \\ \mathbf{C}_2\mathbf{g} \end{bmatrix}, \tag{12}$$

where $\mathbf{g}$ arranged as in (4) denotes the Nyquist(K) FIR filter sequence and $\tilde{\mathbf{h}}$ the sampled amplitude response of the resulting square-root prototype. The product $\mathbf{C}_1\mathbf{g}$ yields the amplitude response for frequencies up to the stopband edge and $\mathbf{C}_2\mathbf{g}$ in the range $\Omega \in [\Omega_s, \pi]$, respectively.

Combining eq. (8) and (12) and rearranging of the weighting matrix $\mathbf{D}$ yields

$$\tilde{\mathbf{h}}_{\mathrm{I}} = \begin{bmatrix} \mathbf{D}_1\sqrt{\mathbf{C}_1\mathbf{g}} \\ \mathbf{D}_2\mathbf{C}_2\mathbf{g} \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{bmatrix}$$

and $\mathbf{D}_1 \in \mathbb{R}^{L_1\times L_1}$, $\mathbf{D}_2 \in \mathbb{R}^{L_2\times L_2}$.

Introducing the inverse transform (7) leads to the desired impulse response vector $\mathbf{h}_{\mathrm{I}}$ in (9), which can be written with $\mathbf{C}^{\mathrm{T}} = \left[\mathbf{C}_1^{\mathrm{T}}\ \mathbf{C}_2^{\mathrm{T}}\right]$ as

$$\mathbf{h}_{\mathrm{I}} = \mathbf{C}_1^{\mathrm{T}}\mathbf{D}_1\sqrt{\mathbf{C}_1\mathbf{g}} + \mathbf{C}_2^{\mathrm{T}}\mathbf{D}_2\mathbf{C}_2\mathbf{g}. \tag{13}$$

Note that the square-root in (11) and (13), resp., is only calculated in the pass- and transition band up to the stopband edge, where $G_0(e^{j\Omega})$ is always positive. Thus, in the resulting response $H_0(e^{j\Omega})$, the stopband edge is moved only slightly to higher frequencies, which results in a nearly unchanged transition bandwidth $b_h \approx b_g$ in comparision to that specified in the original Nyquist(K) filter design.

Since the eigenfilter design is optimal in the least-squares sense, the transfer function $G_0(z)$ exhibits single almost equidistant zeros on the unit circle in the stopband region. Thus, by choosing $H_0(e^{j\Omega}) = G_0(e^{j\Omega})$ in the stopband region, this structure can be retained in the resulting prototype filter, which leads to maximum stopband attenuation.

4.2 Even-length filters

Unlike in the odd-length case, here it is not possible to start with the design of a Nyquist(K) filter, since even-length linear-phase filters cannot satisfy the Nyquist condition (10) due to the even symmetry in their impulse response. However, it is possible to design an even-length lowpass filter $g(n) = g_0(n - S)$, which approximately satisfies the reduced frequency-domain condition

$$G_0\left(e^{j\Omega}\right) + G_0\left(e^{j(\Omega-2\pi/K)}\right) = \begin{cases} 1 & 0 \le \Omega < \frac{2\pi}{K}, \\ \text{arbitr.} & \text{elsewhere.} \end{cases} \tag{14}$$

This can be realized by including this flatness constraint into the objective function of an eigenfilter design [8], which shall be explained in the following. The main idea behind the eigenfilter design is to minimize an objective function, where the minimization process can be expressed as an eigenvalue problem.

The stopband energy can be written with a vector notation for even-length filters similar to (3), (4) as

$$E_s = \int_{\Omega_s}^{\pi} G_0^2(e^{j\Omega})\, d\Omega = \mathbf{g}^{\mathrm{T}}\mathbf{P}\mathbf{g} \tag{15}$$

$$\text{with} \quad \mathbf{P} = \int_{\Omega_s}^{\pi} \mathbf{c}(\Omega)\mathbf{c}^{\mathrm{T}}(\Omega)\, d\Omega \in I\!R^{N/2\times N/2},$$

$$\mathbf{c}(\Omega) = [\cos(S\Omega), \cos((S-1)\Omega), \ldots, \cos((S-N/2+1)\Omega)]^{\mathrm{T}}$$

and should obviously be minimized in order to achieve maximum stopband attenuation.

The deviation from the relaxed condition in (14) can be expressed in terms of an energy E_f as

$$E_f = \int_0^{2\pi/K} \left[1 - G_0\left(e^{j\Omega}\right) - G_0\left(e^{j(\Omega-2\pi/K)}\right)\right]^2 d\Omega$$

and should be zero in the ideal case. Introducing again a notation as in (3), (4) yields the matrix formulation

$$E_f = \mathbf{g}^{\mathrm{T}}\mathbf{Q}\mathbf{g} \tag{16}$$

with

$$\mathbf{Q} = \int_0^{2\pi/K} \left(\mathbf{1} - \mathbf{c}(\Omega) - \mathbf{c}_1(\Omega)\right)\left(\mathbf{1} - \mathbf{c}(\Omega) - \mathbf{c}_1(\Omega)\right)^{\mathrm{T}} d\Omega,$$

where $\mathbf{c}_1(\Omega)$ denotes the length $N/2$ vector of modulated cosines

$$\begin{aligned} \mathbf{c}_1(\Omega) \quad = \quad & [\cos(S(\Omega - 2\pi/K)), \cos(S(\Omega - 2\pi/K) - 1), \\ & \ldots, \cos(S(\Omega - 2\pi/K) - N/2 + 1)]^{\mathrm{T}} \end{aligned}$$

and $\mathbf{1}$ the unity vector of the same length.

The composite objective function to be minimized is now given by [8]

$$\phi = \alpha E_s + (1-\alpha)E_f \stackrel{!}{=} \min.,$$

where α, $0 < \alpha \le 1$ allows a weighting between stopband attenuation and flatness of the amplitude response. With (15) and (16) this results in

$$\phi = \mathbf{g}^{\mathrm{T}}\mathbf{R}\mathbf{g} \quad \text{with} \quad \mathbf{R} = \alpha\mathbf{P} + (1-\alpha)\mathbf{Q}. \tag{17}$$

Since $\mathbf{R}$ is positive definit, Rayleigh's principle can be applied, stating that the vector $\mathbf{g}$, which minimizes ϕ, is the eigenvector corresponding to the smallest eigenvalue λ_0 of $\mathbf{R}$ [8].

The further design of the *desired* square-root Nyquist(K) prototype is again carried out according to eq. (13) with the modifications $L_2 = L/2 - L_1$ and $\mathbf{D}_2 = \mathbf{I}$ due to the even prototype filter length.

5 DESIGN EXAMPLES

As a first example, a prototype filter of length $N = 512$ with $K = 64$, $\alpha = 0.9$ and a transition bandwidth of $b_h \approx 0.03\pi$ is designed, where the number of sampling points is chosen as $L = 1024$.

(a)

(b)

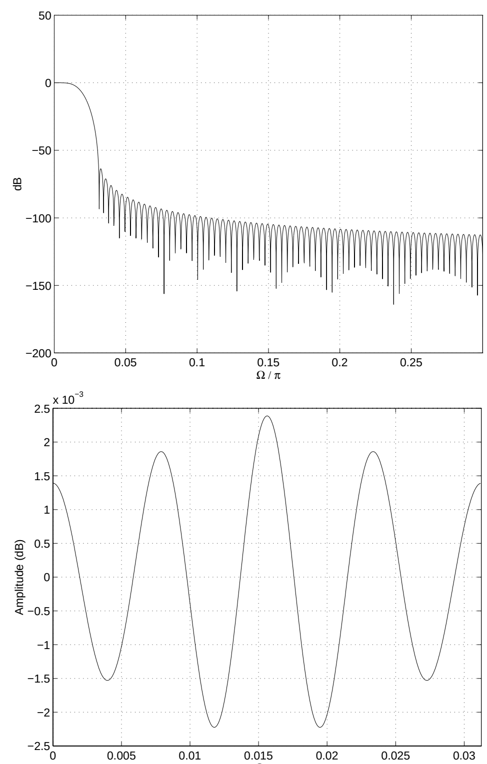

Figure 1: Design example with $N = 512$ and $K = 64$: (a) Magnitude frequency response, (b) Period of the overall transfer function.

Fig. 1 shows the magnitude frequency response and one period of the overall magnitude transfer function $|T(e^{j\Omega})| = \sum_{k=0}^{K-1} \left|H(e^{j(\Omega - k\cdot 2\pi/K)})\right|^2$ for

$0 \leq \Omega < 2\pi/K$. Note that the stopband attenuation is getting larger for increasing Ω, which is useful when processing signals with (mainly) lowpass-shaped power spectra. Such a shape of the stopband response leads to a better suppression of high-energy lowpass aliasing components in the upper subbands.

(a)

(b)

(c)

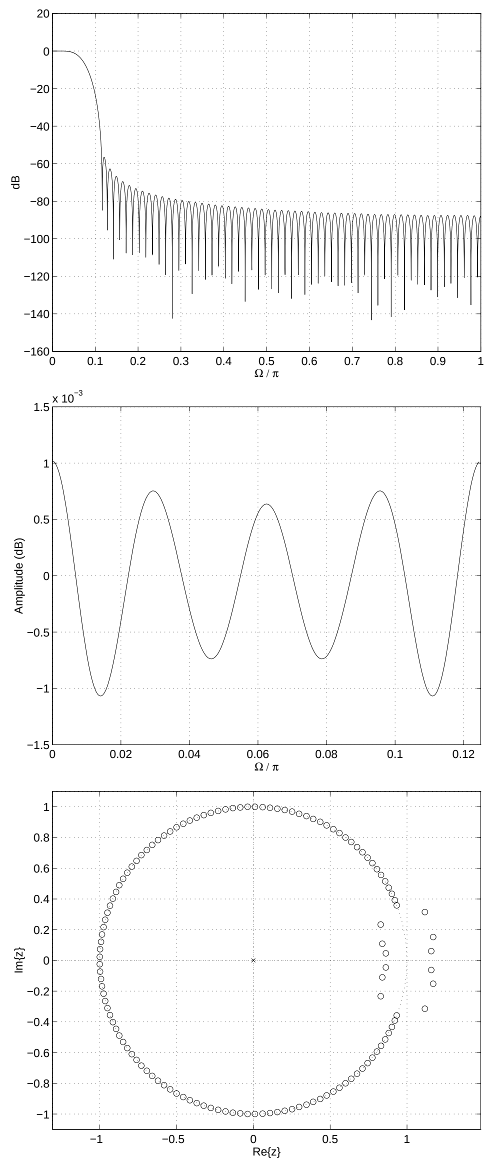

Figure 2: Design example with $N = 129$ and $K = 16$: (a) Magnitude frequency response, (b) Period of the overall transfer function, (c) Zero-pattern.

This can also be observed in the next example, where an odd-length prototype with $N = 129$, $K = 16$, $b_h \approx 0.1\pi$, $L = 1024$ and $\alpha = 0.5$ is designed (Fig. 2). The zero-pattern shows that the single zeros in the stopband region of the unit circle are distributed almost equidistantly, corresponding to minimal stopband energy E_s.

6 CONCLUSION

The design procedure presented in this paper does not depend on a computational expensive optimization routine. This results in a faster and more stable prototype computation, especially for longer impulse responses, where optimization-based approaches often lead to inadequate filters.

Additionally, this method allows an easy implementation. The eigenvector and eigenvalue computation routines for example, which are needed for the first design step, can be found in many numerical packages and libraries.

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