Recursive Volterra Filters with Stability Monitoring

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ABSTRACT

In this paper we describe a sufficient stability condition for a p-th order recursive Volterra filter. Moreover, we show an application of the stability condition to a system identification problem.

1 Introduction

In spite of the great importance of linear filters and system models in a large variety of situations, there are several applications in which they do not perform well. In the presence of multiplicative noise, for example, linear filters performance are unacceptable. For this reason, recently much attention has been drawn to the problem of nonlinear system modeling with a Volterra series expansion [1]. A very popular class of nonlinear filters is the so-called polynomial (or Volterra) filter class. Some successfull applications have been developed in system identification, signal processing, image processing, channel equalization, echo cancellation [2] and telecommunication areas [3] using such nonlinear models. A key point of Volterra filters is that the system output is linearly dependent from the filters' coefficients, and thus relatively simple algorithms for the determination of the coefficients can be developed. One drawback of polynomial filters is, however, that they require a large number of coefficients to characterize a nonlinear process. This problem can be alleviated by using recursive polynomial structures which, just as the linear counterpart, need a lower number of coefficients. However, recursive polynomial filters are inherently unstable, in the sense that it is always possible to find bounded input signals which drive the nonlinear filter to instability. As a matter of fact, not only the stability of a recursive nonlinear filter depends on the filter coefficients, but also the input signal must belong to a certain class to produce a bounded output. This subset of input signals depends only from the recursive filter structure. The notion of input-dependent stability for nonlinear recursive filters was introduced in [5] for bilinear systems, which are perhaps the simplest form of recursive nonlinear models. In [5, 6] input dependent sufficient stability conditions have been reported; in [8] the conditions of [5] have been used to monitor the stability of a bilinear system model in a system identification task. The relation between RLS adaptive bilinear filtering and input dependent stability is reported in [6]. In [4], moreover, the input-dependent stability concept is applied to certain quadratic recursive polynomial filters.

In this paper we extend the results of [4] to a general form of quadratic filter. In Section 2, in fact, we describe a simple sufficient stability condition for recursive second order polynomial filters described by the following p-th order difference equation:

$$y(n) = x(n) + \sum_{i_1=1}^{N_1} h_1(i_1) y(n-i_1) + \sum_{i_1=1}^{N_2}\sum_{i_2=1}^{N_2} h_2(i_1,i_2) y(n-i_1) y(n-i_2) + \cdots + \sum_{i_1=1}^{N_p} \cdots \sum_{i_p=1}^{N_p} h_p(i_1 \cdots i_p) y(n-i_1) \cdots y(n-i_p) \quad (1)$$

The paper is organized as follows: in Section 2 the sufficient condition is described, and in Section 3 some experimental results, namely a verification of the stability conditions and an application to a system identification task, are reported.

2 Input-dependent stability conditions

For simplicity, let us introduce the time-invariant operator q^{-1} such that

$$\sum_{i_1=1}^{N_1} h_1(i_1) y(n-i) = \left(\sum_{i_1=1}^{N_1} h_1(i_1) q^{-i} \right) y(n)$$

Moreover, let us call p_i the zeros of the polynomial

$$\sum_{i=0}^{N_1} b_i z^i \quad (2)$$

where $b_i = -h_1(N_1-1)$ for $i = 0 \cdots (N_1-i)$ and $b_{N_1} = 1$ and introduce the following terms:

$$\gamma_1 = \prod_{i=1}^{N_1} (1 - |p_i|) \quad (3)$$

$$\gamma_2 = -\sum_{i_1}\sum_{i_2} |h_2(i_1, i_2)| \quad (4)$$

$$\vdots$$

$$\gamma_p = -\sum_{i_1}\cdots\sum_{i_p} |h_p(i_1 \cdots i_p)| \quad (5)$$

We now prove a sufficient stability condition for the recursive system (1).

Theorem 1 *Assume that the system (1) is initially at rest, namely $x(n) = y(n) = 0$ for $n < 0$. A stability condition for every bounded input signal to produce a bounded output signal for the recursive model described by (1) is the following:*

$$\begin{cases} |p_i| < 1 & \text{for } i = 1, 2, \cdots, N_1 \\ |x(n)| \leq M_x & \forall n \text{ and with } M_x = \sum_{i=1}^{p} \gamma_i M_y^i \end{cases}$$

where M_y is the positive real root of the following polynomial

$$\sum_{i=1}^{p} i\gamma_i z^{i-1} \quad (6)$$

and p is the filter order in (1). If this condition is met, moreover, the output signal is bounded by M_y.

Proof With the q^{-i} operator described above, the recursive quadratic system reported in (1) can be rewritten in the following way:

$$\left(1 - \sum_{i_1=1}^{N_1} h_1(i_1) q^{-i}\right) y(n) =$$

$$x(n) + \sum_{i_1=1}^{N_2}\sum_{i_2=1}^{N_2} h_2(i_1, i_2) y(n - i_1) y(n - i_2) + \cdots +$$

$$+ \sum_{i_1=1}^{N_p}\cdots\sum_{i_p=1}^{N_p} h_p(i_1 \cdots i_p) y(n - i_1) \cdots y(n - i_p) \quad (7)$$

and, by defining

$$k(n) = x(n) + \sum_{i_1=1}^{N_2}\sum_{i_2=1}^{N_2} h_2(i_1, i_2) y(n - i_1) y(n - i_2) +$$

$$+ \cdots +$$

$$+ \sum_{i_1=1}^{N_p}\cdots\sum_{i_p=1}^{N_p} h_p(i_1 \cdots i_p) y(n - i_1) \cdots y(n - i_p) \quad (8)$$

one can write

$$\left(1 - \sum_{i=1}^{N_1} h_1(i_1) q^{-i}\right) y(n) = k(n) \quad (9)$$

or, expressing (10) in terms of the zeroes of (2), p_i, we can write

$$y(n) = \left\{ \prod_{i=1}^{N_1} \left(1 - p_i q^{-1}\right)^{-1} \right\} k(n) \quad (10)$$

According to Lee-Mathews, considering a first order system, with $N_1 = 1$, we have $y(n) = \left(1 - p_i q^{-1}\right)^{-1} k(n)$ or, by successive substitutions,

$$y(n) = \left\{ \sum_{l=0}^{\infty} p_i^l q^{-l} \right\} k(n) \quad (11)$$

Suppose now that (11) is true for all the system orders from 1 to $(N_1 - 1)$ and let us see if it holds for the N_1 system order. The N_1-th order system can be viewed as the cascade of a first order system with a pole p_{N_1} and a $(N_1 - 1)$-th order system with poles $p_1 \cdots p_{N_1 - 1}$. Hence

$$y(n) = \left\{ \sum_{l=0}^{\infty} p_{N_1}^l q^{-l} \right\} \left\{ \prod_{i=1}^{N_1 - 1} \sum_{l=0}^{\infty} p_i^l q^{-l} \right\} k(n) \quad (12)$$

or

$$y(n) = \left\{ \prod_{i=1}^{N_1} \sum_{l=0}^{\infty} p_i^l q^{-l} \right\} k(n) \quad (13)$$

Substituting (9) in (13), it follows that

$$y(n) = \left\{ \prod_{i=1}^{N_1} \sum_{l=0}^{\infty} p_i^l q^{-l} \right\} x(n) +$$

$$+ \left\{ \prod_{i=1}^{N_1} \sum_{l=0}^{\infty} p_i^l q^{-l} \right\} \sum_{i_2=1}^{N_2}\sum_{i_2=1}^{N_2} h_2(i_1, i_2) y(n - i_1) y(n - i_2) +$$

$$+ \cdots +$$

$$+ \left\{ \prod_{i=1}^{N_1} \sum_{l=0}^{\infty} p_i^l q^{-l} \right\} \sum_{i_1=1}^{N_p}\cdots\sum_{i_p=1}^{N_p} h_p(i_1 \cdots i_p) \cdot$$

$$\cdot y(n - i_1) \cdots y(n - i_p) \quad (14)$$

and, from (14)

$$|y(n)| \leq \left\{ \prod_{i=1}^{N_1} \sum_{l=0}^{\infty} |p_i|^l q^{-l} \right\} |x(n)| +$$

$$+ \left\{ \prod_{i=1}^{N_1} \sum_{l=0}^{\infty} |p_i|^l q^{-l} \right\} \sum_{i_1=1}^{N_2}\sum_{i_2=1}^{N_2} |h_2(i_1, i_2| \cdot$$

$$\cdot |y(n - i_1)||y(n - i_2)| + \cdots + \left\{ \prod_{i=1}^{N_1} \sum_{l=0}^{\infty} |p_i|^l q^{-l} \right\} \cdot$$

$$\cdot \sum_{i_1=1}^{N_p}\cdots\sum_{i_p=1}^{N_p} |h_p(i_1 \cdots i_p)||y(n - i_1)| \cdots |y(n - i_p)| \quad (15)$$

Let us now suppose that $|x(n)| \leq M_x$ for every n. Clearly, since the filter is initially at rest, $|y(n)| \leq M_y$ for $n < 0$. Furthermore, let us suppose that $|y(n - i)| \leq M_y$ for each $i \geq 1$. Then

$$|y(n)| \leq \left\{ \prod_{i=1}^{N_1} \sum_{l=0}^{\infty} |p_i|^l \right\} M_x +$$

$$+\left\{\prod_{i=1}^{N_1}\sum_{l=0}^{\infty}|p_i|^l\right\}\sum_{i_1=1}^{N_2}\sum_{i_2=1}^{N_2}|h_2(i_1,i_2|M_y^2+\cdots+$$

$$+\left\{\prod_{i=1}^{N_1}\sum_{l=0}^{\infty}|p_i|^l\right\}\sum_{i_1=1}^{N_p}\cdots\sum_{i_p=1}^{N_p}|h_p(i_1\cdots i_p)|M_y^p \quad (16)$$

Since $|p_i|<1$, we have

$$\sum_{l=0}^{\infty}|p_i|^l=(1-|p_i|)^{-1} \quad (17)$$

and, therefore, from the definition of the γ_i given above, we can rewrite (16) as follows:

$$|y(n)|\le\frac{1}{\gamma_1}M_x-\frac{\gamma_2}{\gamma_1}M_y^2-\cdots-\frac{\gamma_p}{\gamma_1}M_y^p \quad (18)$$

In order to satisfy the above condition, we can impose that the right term of (18) be bounded by M_y. In this way, from (18) we get

$$M_x\le\sum_{i=1}^{p}\gamma_i M_y^i \quad (19)$$

To find an M_y which satisfy (19), we should find the positive extrema of the function $\mathcal{F}=\sum_{i=1}^{p}\gamma_i M_y^i$ which can be found by setting to zero its first derivative. That is,

$$\frac{d\mathcal{F}}{dM_y}=\sum_{i=1}^{p}i\gamma_i M_y^{i-1}=0 \quad (20)$$

It is important to remark that, due to the Descarte's rule of sign, the polynomial (20) has only a positive real root. Let us call $\overline{M_y}$ this root, which can be found with numerical algorithms [10]. The corresponding extrema is the bound on x(n) we are looking for, and it is given by $M_x=\sum_{i=1}^{p}\gamma_i\overline{M_y}^i$.

In conclusion, we can say that if the input signal $x(n)$ is bounded by M_x for every n, then the output signal $y(n)$ is bounded by M_y for every n. □

Remark. The stability condition can be shown not to be necessary for stability by showing that there exist stable recursive polynomial filters that do not satisfy the input bound. This will be illustrated in the next Section.

3 Experimental results

3.1 Step response of a nonlinear filter

This stability condition has been implemented and extensively tested in a number of situations. In Fig.1 the step response of the third order filter

$$y(n)=x(n)+0.5y(n-1)-0.06y(n-2)+$$
$$+0.02y^2(n-1)+0.03y^3(n-1) \quad (21)$$

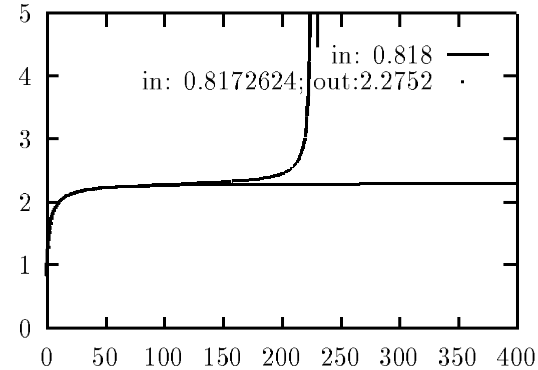

Figure 1: Step responses of (21) at different input amplitudes.

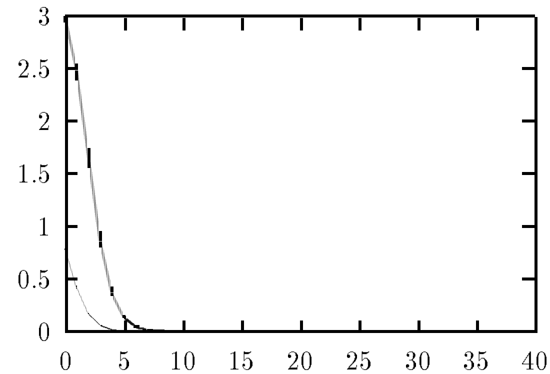

Figure 2: Impulse responses of (21) with input amplitudes of 0.8 and 3, respectively.

at different signal amplitudes is shown. The filter's coefficients give the following values for the parameters introduced in Section 2: $\gamma_1=0.56;\gamma_2=-0.02;\gamma_3=-0.03$ and therefore, we get $M_y=2.282095$ and $M_x=0.8172624$. In this example, if the value of the input amplitude is equal to M_x, the output is bounded. But, as soon as the input signal exceeds this bound, instability arises. However, if we consider the impulse response of the same nonlinear system, reported in Fig.2, we can see that system is stable even if the input signal greater than M_x. This should be compared with the above *Remark*.

3.2 An identification experiment with stability monitoring

Given a desired response signal $d(n)$ and an input signal $s(n)$, we estimate $d(n)$ adaptively using output-error LMS. Introducing the coefficient vector $\Theta(n)$ and the signal vector $\Phi(n)$, both at the time instant n and defined by (22) and (23), respectively,

$$\Theta^T(n)=|1,h_1(1),\cdots,h_1(N_1),h_2(1,1),$$
$$\cdots,h_2(N_2,N_2),\cdots,h_p(1,\cdots,1),\cdots,h_p(N_p,\cdots,N_p)| \quad (22)$$

$$\Phi^T(n)=|x(n),\hat{d}(n-1),\cdots,\hat{d}(n-N_1),\cdots,\hat{d}(n-1)^2,$$

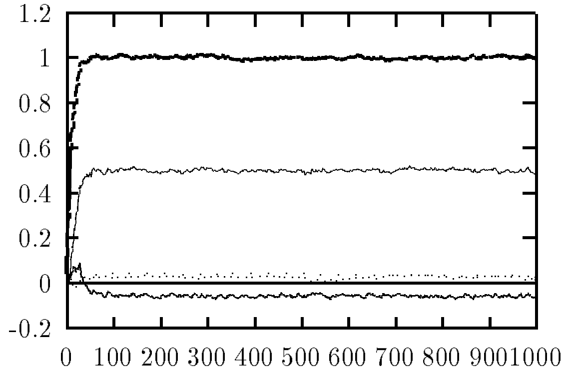

Figure 3: Coefficients trajectory in the system identification experiment described in Section 3.2

$$\cdots, \hat{d}(n-N_2)^2, \cdots, \hat{d}(n-1)^p, \cdots, \hat{d}(n-N_p)^p| \quad (23)$$

the adaptive recursive polynomial filtering algorithm with output-error LMS can be summarized by the following equations:

$$e(n) = d(n) - \hat{d}(n) \quad (24)$$

$$\hat{d}(n) = \Theta(n)^T \Phi(n) \quad (25)$$

$$\Phi(n+1) = \Phi(n) + \mu\Phi(n)e(n) \quad (26)$$

In (24), $e(n)$ is the output error and $\hat{d}(n)$ is the output of the adaptive filter. The algorithm described in (24)-(26) is not guaranteed to be stable because the output $\hat{d}(n)$ may grow without bound during adaptation and therefore the system (1) must be continuosly monitored for stability using **Theorem 1**. The scheme we used to overcome the stability problems with recursive polynomial filters is the approximated one proposed in [8]: after each coefficient update, the stability conditions described in Theorem 1 are checked to see if stability is guaranteed. If not, the amount of adjustment for the coefficient vector is reduced in the hope that the resulting update will pass the stability test. The result of the identification of an unknown, recursive third-order system whose difference equation is described in (21) is reported in fig.2, which shows the coefficients trajectories versus the number of iterations. In this experiment, the input signal $x(n)$ was a white, zero-mean and pseudorandom Gaussian noise with a variance equal to 0.2. Another white, zero-mean, pseudorandom Gaussian noise with a variance equal to 0.01 was added to the output of the unknown system. The results presented in Fig.2 represent ensemble averages over 5 independent runs.

4 Conclusions

The sufficient condition proved in Section 2 requires the computation of the zeros of (2) and the computation of the terms γ_i defined in (3)-(5). Since the zeros of a polynomial can be computed adaptively [9], the stability condition could be used for adaptive polynomial filters. The stability condition has been experimentally verified and applied to a nonlinear system identification task for monitoring its stability and some experimental results have been described. Currently, we are concerned with the development of improved techniques for adaptive nonlinear system identification with stability monitoring and with the design of stable recursive systems.

Acknowledgments

This work was partially supported by the Esprit Project NOBLESSE, Nr. 20229

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