COMPARISON OF DOA ESTIMATION PERFORMANCE FOR VARIOUS TYPES OF SPARSE ANTENNA ARRAY GEOMETRIES

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ABSTRACT

Three subclasses of geometries for nonuniform linear antenna arrays with a fixed number of sensors are compared in the sense of maximum possible direction-of-arrival (DOA) estimation accuracy. Cramer-Rao bound analysis is applied to compare the optimal accuracy for each geometry under some fixed source environment. Actual DOA estimation simulations, obtained by recently-introduced algorithms, are used to demonstrate the applicability of Cramer-Rao bound analysis for DOA estimation in these cases. We show that previous attempts to maximise the number of contiguous correlation lags and to avoid missing lags in certain array geometries does not necessarily lead to an improvement in DOA estimation performance.

1 INTRODUCTION

When the number of antenna sensors available for a linear array is limited, the problem of optimum array geometry naturally arises. For linear arrays, solutions to this problem belong to the class of nonuniformly-spaced linear arrays (NLA's), also known as sparse or aperiodic arrays, and several different approaches currently exist which seek the "best" design [1].

The class of sparse arrays introduced by Moffet [2] to achieve maximum resolution for a given number of antenna elements are known as "minimum-redundancy" arrays. Rather than retaining Moffet's original nomenclature of "restricted" and "general" minimum-redundancy arrays, this paper discusses comparative DOA estimation accuracy for three proposed array subclasses.

Let the NLA geometry be specified by the sensor positions d_i, $(i = 1, \dots, M)$, set $d_1 = 0$ for convenience, and let d be the greatest common divisor of the difference set

$$\mathcal{D} = \{d_i - d_j \mid i, j = 1, \dots, M;\ i > j\}. \tag{1}$$

Denote the maximum inter-element distance (array aperture) by $d(M_\alpha - 1)$.

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The first subclass we consider are the so-called fully-augmentable arrays, which have the property that all intermediate distances are realised; *ie.* given the sequence of natural numbers $\kappa = 1, \dots, M_\alpha - 1$, we have $\kappa d \in \mathcal{D}$. Optimum-lag NLA's have a complete and nonredundant set of correlation lags. Since these exist only in a very limited number of cases ($M \leq 4$) [2], some of the correlation lags present in the difference set of a fully-augmentable array are generally duplicated, and this redundancy (in the form of the number of redundant lags R) is often minimised. We shall reserve the use of the term *minimum-redundancy* arrays for this first subclass only.

The second subclass we consider was introduced by Moffet [2] under the name "general" arrays. These arrays maximise $N_{\max}$, the greatest multiple of the unit spacing d such that all lags up to $N_{\max}$ (inclusive) are present, regardless of the number of redundant lags R or missing lags (gaps) G. In other words, the smallest gap has length $(N_{\max} + 1)$ in units of d. We shall call this subclass *maximum-contiguous-lag* arrays.

The third subclass [3] presents each lag precisely once:

$$d_i - d_j \neq d_k - d_\ell \quad \forall\, i, j, k, \ell = 1, \dots, M;\ i > j,\ k > \ell. \tag{2}$$

Not all possible lags are present in this subclass of partially-augmentable arrays; for a given M, the number of missed lags is minimised, regardless of their position. We shall call this subclass *minimum-gaps* arrays.

These three subclasses may be succinctly defined as follows:

1. minimum-redundancy arrays minimise R such that $G = 0$,
2. maximum-contiguous-lag arrays maximise $N_{\max}$ irrespective of R and G, and
3. minimum-gaps arrays minimise G such that $R = 0$.

For a given number of sensors M, these subclasses propose correspondingly "optimal" antenna geometries with differing aperture $(M_\alpha - 1)$, redundancy (R), incompleteness (G) and contiguous completeness $(N_{\max})$. Since each of these parameters indirectly affects DOA

antenna geometry	M_α	R	G	$N_{\max}$	redundancies/gaps	min-red	min-gaps	max-ctgs-lag
$\boldsymbol{d_1} = [0,1,2,6,9] \equiv [0,3,7,8,9]$	10	1	0	9	redundancy $= \{1\}$	$\surd$		$\surd$
$\boldsymbol{d_2} = [0,1,4,7,9] \equiv [0,2,5,8,9]$	10	1	0	9	redundancy $= \{3\}$	$\surd$		$\surd$
$\boldsymbol{d_3} = [0,1,4,9,11] \equiv [0,2,7,10,11]$	12	0	1	5	gap $= \{6\}$		$\surd$	
$\boldsymbol{d_4} = [0,2,7,8,11] \equiv [0,3,4,9,11]$	12	0	1	9	gap $= \{10\}$		$\surd$	$\surd$
$\boldsymbol{d_5} = [0,4,5,7,13] \equiv [0,6,8,9,13]$	14	0	3	9	gaps $= \{10,11,12\}$			$\surd$

Table 1: All minimum-redundancy, minimum-gaps and maximum-contiguous-lag arrays for $M = 5$ with their main characteristics.

estimation accuracy in a complicated fashion, we have used Cramer-Rao bound analysis [4, 5] to compare the optimal accuracies for various competing geometries in identical signal environments. We also provide examples of stochastic simulations for DOA estimation algorithms [6, 7] for both fully- and partially-augmentable arrays which have been proven to deliver accuracy reasonably close to the Cramer-Rao bound.

2 BACKGROUND

For a signal environment consisting of m uncorrelated sources, the deterministic signal covariance matrix $\mathcal{R}_s$ of the outputs of the array is the sum of the m dyads $B(\theta)\, B^H(\theta)$, weighted by the source powers p_i:

$$\mathcal{R}_s \;=\; \sum_{i=1}^{M} p_i\, B(\theta_i)\, B^H(\theta_i) \tag{3}$$

where the "steering vector"

$$B(\theta_i) = \left[1, \exp\left(i2\pi\frac{d_2}{\lambda}\sin\theta_i\right), \ldots, \exp\left(i2\pi\frac{d_M}{\lambda}\sin\theta_i\right)\right]^T \tag{4}$$

is defined by the particular plane-wave DOA θ_i, measured with respect to broadside.

In the presence of additive white noise of power σ, we have

$$\mathcal{R} = \mathcal{R}_s + \sigma I_M . \tag{5}$$

This covariance matrix is Hermitian with constant diagonal elements, and off-diagonal elements

$$\mathcal{R}_{\kappa\ell} \;=\; \sum_{j=1}^{M} p_j \exp\left[2\pi i\frac{(d_\kappa - d_\ell)}{\lambda}\sin\theta_j\right] \quad \begin{matrix}\kappa \neq \ell \\ \kappa, \ell = 1, \ldots, M.\end{matrix} \tag{6}$$

We make the typical assumption $d = \lambda/2$, and must first address the important question of identifiability and ambiguity of these arrays. Recent algorithms [8] applied to examples from the above three array subclasses have demonstrated that *all* these sparse array types, including fully-augmentable arrays, are ambiguous, *ie.* admit the possibility of a rank-deficient signal manifold $B(\theta)$ under specific source environments.

Consider the optimum-lag array $\boldsymbol{d} = [0,1,4,6]$, for example. The set of three DOA's $\sin\boldsymbol{\theta} = \{-2/3, 0, 2/3\}$ form an ambiguous (rank 2) set. Naturally, the standard MUSIC algorithm fails to cope with this ambiguity. That is, MUSIC($\mathcal{R}$) for any two of these three sources results in three peaks. Fortunately however, the augmented covariance matrix T defined by the specified lags is unambiguous.

Thus we may define the following regimes:

Where $m < N_{\max}$, the non-ambiguity of the Carathéodory representation [9] for the fully-specified augmented Toeplitz covariance matrix T may be used for true DOA identification. For $1 < m < M$, the possible failure of standard MUSIC applied to the M-variate rank-deficient matrix $\mathcal{R}$ should be taken into account.

Where $N_{\max} \leq m \leq \frac{1}{2}M(M-1) - R$, the conditions for the non-ambiguity of the specified covariance lags are as yet unknown. However, rank-deficiency of the Fisher (information) matrix [4, 5] define the identifiability conditions.

Where $m > \frac{1}{2}M(M-1) - R$, there are no possible conditions for non-ambiguity since the Fisher matrix is always rank deficient.

3 RESULTS OF CRB COMPARISONS

Comparative analysis of the optimal DOA estimation accuracy have been obtained for arrays with $M = 5$ sensors, where an exhaustive search has been conducted to find all possible competitive geometries from the three subclasses (see Table 1).

The results are illustrated in Figure 1 where the source separation $\Delta\omega$ is uniform and fixed, while the number of sources m varies; and in Figure 2 where the spatial separation varies for a fixed number of sources ($m = 8$). (Throughout this paper, we measure spatial separation ω in units of $2\pi d/\lambda$. Also, note that while the maximum CRB may approach infinity, its physical interpretation as maximum DOA estimation RMSE is limited in value to $1/\sqrt{3}$.)

We see that in these situations, the maximum-contiguous-lag array $\boldsymbol{d_5}$ and the minimum-gaps arrays $\boldsymbol{d_3}$ and $\boldsymbol{d_4}$ are mostly superior to the minimum-redundancy arrays $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$. Moreover, the minimum-redundancy arrays are also somewhat inferior in that

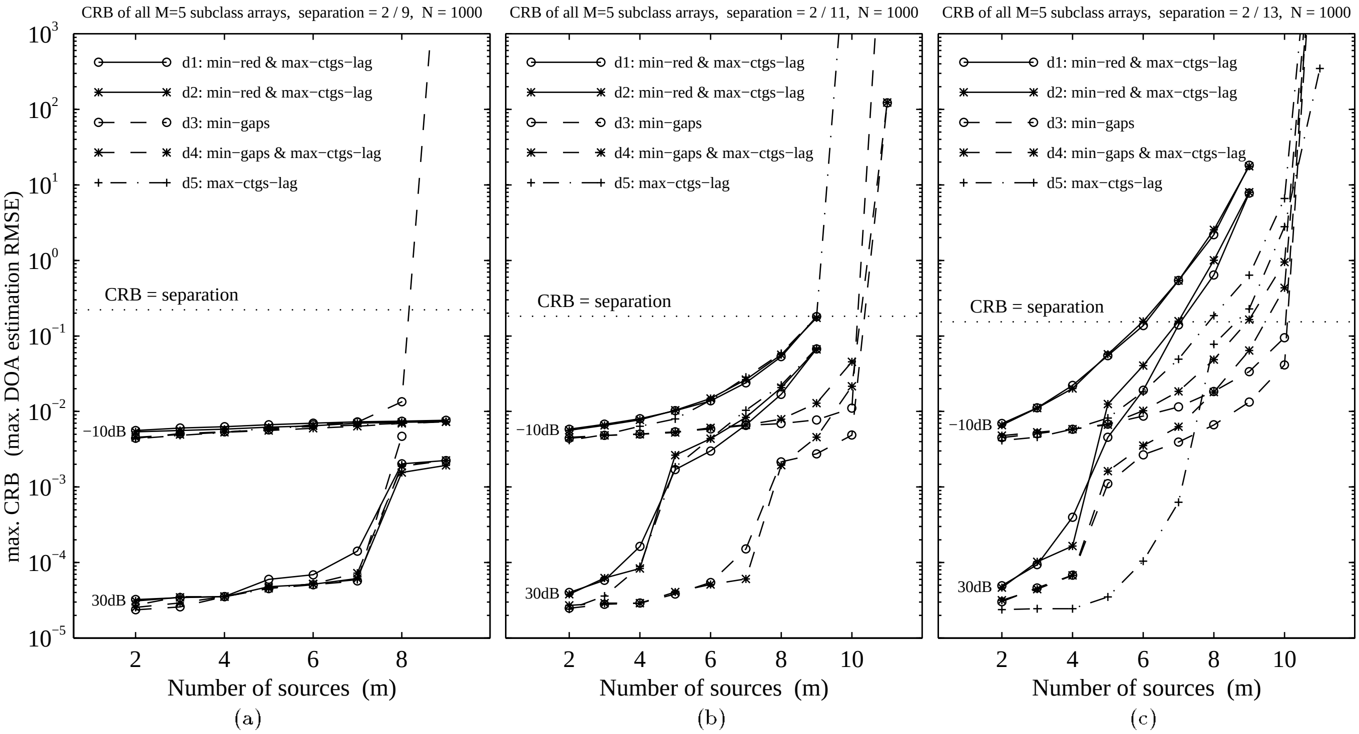

Figure 1: Sample CRB comparison between the five NLA geometries listed in Table 1 for separations (a) $\Delta w = 2/9$ (the Rayleigh limit for $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$), (b) $\Delta w = 2/11$ (the Rayleigh limit for $\boldsymbol{d_3}$ and $\boldsymbol{d_4}$), and (c) $\Delta w = 2/13$ (the Rayleigh limit for $\boldsymbol{d_5}$). Results are for nominal figures of $N = 1000$ snapshots and SNR $= -10$ dB and 30 dB.

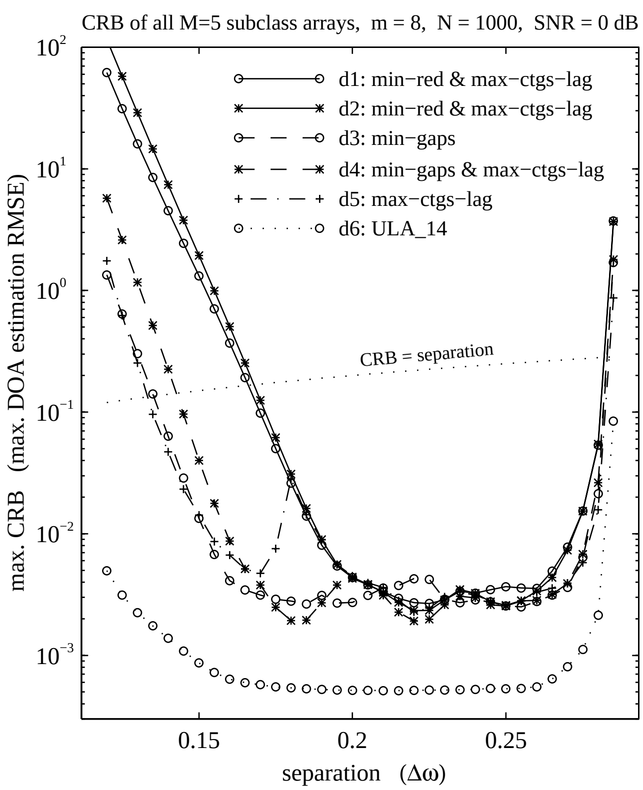

Figure 2: Sample CRB comparison between the five NLA geometries listed in Table 1 and one ULA geometry for $m = 8$ sources at varying separations Δw.

they cannot detect the maximum number of sources $m_{\max} = \frac{1}{2}M(M{-}1) = 10$, since their limit is $m_{\max} - R = 9$. Note that in some situations, minimum-gaps arrays are superior to maximum-contiguous-lag arrays, regardless of the greater aperture of the latter. However, when the source separation Δw is decreased below the Rayleigh resolution limit $\Delta w_{\text{crit}} = 2/(M_\alpha{-}1)$ for the minimum-gaps array, the maximum-contiguous-lag array $\boldsymbol{d_5}$ outperforms the others mainly because of its greater aperture.

It is interesting to compare the optimal accuracies for the two minimum-gaps arrays ($\boldsymbol{d_3}$ and $\boldsymbol{d_4}$) which differ only by their (single) gap. Though the array $\boldsymbol{d_3}$ has an "earlier" gap than $\boldsymbol{d_4}$ (indeed $N_{\max}^{d_3} < m < N_{\max}^{d_4}$), the optimal performance of $\boldsymbol{d_3}$ is generally better than that of $\boldsymbol{d_4}$, depending on source separation. This difference is due to different interactions between the array beam patterns and the varying source separations.

Finally, comparison with a 14-element uniformly-spaced linear array (ULA) makes clear the tradeoff between number of antenna sensors and optimal accuracy.

Note that the maximum number of identifiable sources is the same for all of these nonredundant geometries ($m_{\max} = 10$), including the maximum-contiguous-lag ones ($\boldsymbol{d_4}$ and $\boldsymbol{d_5}$). Naturally, the possible redundancy of the maximum-contiguous-lag arrays potentially reduces their $m_{\max}$ compared with minimum-gaps (nonredundant) geometries.

	estimator	max RMSE	max bias
$\boldsymbol{d_1}$	root-MUSIC($\overline{T}_{GS}$)	0.0031	0.0002
	MUSIC($\overline{T}_{GS}$)	0.0031	0.0002
	CRB	0.0027	0
$\boldsymbol{d_2}$	root-MUSIC($\overline{T}_{GS}$)	0.0025	0.0002
	MUSIC($\overline{T}_{GS}$)	0.0025	0.0002
	CRB	0.0023	0
$\boldsymbol{d_3}$	root-MUSIC(T_{mME})	0.0029	0.0041
	MUSIC(T_{mME})	0.0029	0.0041
	CRB	0.0043	0
$\boldsymbol{d_4}$	root-MUSIC(T_{mME})	0.0019	0.0025
	MUSIC(T_{mME})	0.0019	0.0025
	CRB	0.0019	0
$\boldsymbol{d_5}$	root-MUSIC(T_{mME})	0.0027	0.0032
	MUSIC(T_{mME})	0.0027	0.0032
	CRB	0.0023	0

Table 2: Stochastic DOA estimation statistics for the same arrays as in Table 1, conducted over 1000 trials. For details of the estimators $\overline{T}_{GS}$ and T_{mME}, see [6, 7].

4 STOCHASTIC SIMULATIONS

For DOA estimation in practice, we may assume that sufficient statistics are available in the form of the direct data covariance (DDC) matrix $\hat{\mathcal{R}}$. This is obtained by sample averaging over the set of N independent training vectors ("snapshots"), originating from a complex Gaussian distribution $\mathcal{CN}(M, 0, \mathcal{R})$.

Recently-introduced techniques based upon matrix augmentation provide improved performance for DOA estimation in fully-augmentable arrays [6]; while for partially-augmentable arrays, maximum-entropy positive-definite Toeplitz matrix completion has been developed [7] as part of a new DOA estimation algorithm.

Since these two papers already discuss the relationship of the Cramer-Rao bound to achievable estimation performance for these algorithms, we limit ourselves here to a single typical simulation. Table 2 shows the maximum bias and root-mean-square values obtained by 1000 computer trials, each of $N = 1000$ snapshots for the five competitive geometries $\boldsymbol{d_1}, \ldots, \boldsymbol{d_5}$ with $m = 8$ sources at spatial separation $\Delta\omega = 0.22$ and SNR = 0 dB. We see that the actual accuracy is in satisfactory agreement with the Cramer-Rao bound.

It is worth mentioning that for separation $\Delta\omega = 0.18$, the ME completion algorithm [7] delivers accuracies close to the Cramer-Rao bound for both minimum-gaps arrays $\boldsymbol{d_3}$ and $\boldsymbol{d_4}$, while for the maximum-contiguous-lag arrays $\boldsymbol{d_5}$, this approach leads to an asymptotically-biased solution. Even this complication agrees with the huge loss in optimal accuracy predicted for this situation by Cramer-Rao bound analysis.

5 SUMMARY

We have provided comparisons of the greatest attainable DOA estimation accuracy which demonstrate that for a fixed number of antenna sensors, minimum-gaps (specifically, nonredundant) arrays have the greatest number of potentially-identifiable sources. These arrays exhibit superior DOA estimation accuracy, except for separations that are below the Rayleigh resolution limit where the maximal-aperture geometries seem superior.

We have shown that the position of the missing lags does not necessarily degrade estimation accuracy, and that an attempt to maximise the number of contiguous lags may not lead to an improvement in accuracy.

Stochastic simulation results provided for recently-developed DOA estimation algorithms for both fully- and partially-augmentable arrays show that the results of comparative Cramer-Rao bound analysis are valid for practically-attainable accuracies.

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