EIGENVECTOR PEELING APPROACH TO COHERENT MULTIPLE SOURCE LOCATION PROBLEM

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ABSTRACT

We propose a novel preprocessing scheme, referred to as *vector peeling*, as an alternate to the conventional spatial smoothing for solving the multiple source location problem involving coherent sources or a rank deficient source covariance matrix. The essence of the technique is to preprocess the signal subspace eigenvectors rather than the covariance matrix as in spatial smoothing. It is shown by analysis and computer simulations that these two approaches are related, and that vector peeling slightly outperforms spatial smoothing when employed with the MUSIC-type DOA estimators. In certain instances, vector peeling offers advantages in terms of computational simplicity and flexibility. The latter is especially true with eigenstructure DOA estimators in adaptive estimation problems, i.e., when the signal subspace eigenvectors are updated using fast adaptive algorithms.

1. INTRODUCTION

The majority of the popular eigenstructure direction of arrival (DOA) estimation methods such as MUSIC [1], Min-Norm [2], and others, are known to perform poorly when the sources are coherent or highly correlated. Spatial smoothing [3] has emerged as a viable technique to combat the problem of source covariance matrix degeneracy in coherent multiple source location. This technique has attracted considerable attention in literature in the recent decade. It has a drawback in that it does not lend itself easily when employed with eigenstructure DOA estimators. In spatial smoothing once the subarray dimension is chosen, it is difficult to enlarge or reduce subarray size as it involves recomputation of eigenvectors/eigenvalues. In order to provide such flexibility, it is more logical to preprocess not the sample covariance matrix but directly the signal subspace eigenvectors [2], [4]. We show how this can be done by means of a process referred to as *vector peeling*. We introduce this process and revisit the conventional spatial smoothing technique. Then, it is shown that spatial smoothing can be reformulated as a eigenvector peeling process. It is demonstrated that these two techniques are theoretically similar but computationally different and, therefore, represent the alternative solutions to coherent multiple source location problem. After that, we develop vector peeling algorithm for location of coherent sources. Though the simplest forward–only peeling and smoothing algorithms are considered, all results can be easily extended to the forward–backward case. The computational loads of vector peeling algorithm are compared with that of spatial smoothing technique and some possible computational advantages of vector peeling are established for signal subspace updating algorithms. Finally, simulation results are given, demonstrating that vector peeling slightly outperforms spatial smoothing for MUSIC-type estimators in coherent source scenarios.

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2. VECTOR PEELING IN RELATION TO SPATIAL SMOOTHING

Define vector peeling as a simple process of generating reduced length, overlapping vectors from a given vector as follows. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is a given $n \times 1$ vector, vector peeling operator $\mathrm{VP}(\cdot)$ produces the $m \times 1$ vector:

$$\mathrm{VP}(\mathbf{x}, i, m) = (x_i, x_{i+1}, \ldots, x_{i+m-1})^T \tag{1}$$

where $m \leq n$, $(\cdot)^T$ denotes the transpose. Thus, the process of vector peeling can produce $k = n - m + 1$ different $m \times 1$ vectors $\mathrm{VP}(\mathbf{x}, i, m)$, $\quad i = 1, 2, \ldots, n - m + 1$ from the underlying vector $\mathbf{x}$.

Define the $m \times n$ matrix $\mathbf{I}_{(i,m)}$ to be a matrix that has first $i-1$ zero columns and last $n-m-i+1$ zero columns, and an $m \times m$ identity matrix $\mathbf{I}_m$ in the middle:

$$\mathbf{I}_{(i,m)} = [0_1, \ldots, 0_{i-1}, \mathbf{I}_m, 0_{i+m}, \ldots, 0_n] \tag{2}$$

where 0_l denotes lth zero column. With this notation, (1) can be rewritten as:

$$\mathrm{VP}(\mathbf{x}, i, m) = \mathbf{I}_{(i,m)}\mathbf{x} \tag{3}$$

Let us now revisit the conventional spatial smoothing technique and show how it is related to vector peeling.

Assuming that q $(q < n)$ narrowband plane wave sources impinge on the equispaced linear array of n sensors from directions $\theta_1, \theta_2, \ldots, \theta_q$ and that the additive noise is zero-mean random process independent from sensor to sensor, the $n \times n$ covariance matrix of array outputs can be written as [1], [2]:

$$\mathbf{R} = \mathbf{A}\mathbf{S}\mathbf{A}^H + \sigma^2\mathbf{I}_n \tag{4}$$

where

$$\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_q)]$$

is the $n \times q$ direction matrix, $\mathbf{a}(\theta_i)$ denotes the direction vector of ith source, $\mathbf{S}$ is the $q \times q$ source covariance matrix, σ^2 denotes the noise variance, and $(\cdot)^H$ denotes the Hermitian transpose. It is well known [3] that when some of the sources are mutually coherent, the source covariance matrix becomes singular and the eigenstructure algorithms of DOA estimation are no longer applicable. The spatial smoothing preprocessing scheme [3] restores the rank of source covariance matrix by partitioning the array into k overlapped subarrays with dimension $m = n - k + 1$ $(m > q)$ and by averaging the subarray output covariance matrices. The $m \times m$ output covariance matrix $\mathbf{R}^{\{i\}}$ of the ith subarray can be represented as:

$$\mathbf{R}^{\{i\}} = \mathbf{I}_{(i,m)}\mathbf{R}\mathbf{I}_{(i,m)}^T \tag{5}$$

and the $m \times m$ spatially smoothed covariance matrix as:

$$\tilde{\mathbf{R}} = \frac{1}{k}\sum_{i=1}^{k}\mathbf{R}^{\{i\}} = \frac{1}{k}\sum_{i=1}^{k}\mathbf{I}_{(i,m)}\mathbf{R}\mathbf{I}_{(i,m)}^{T} \qquad (6)$$

Write the eigendecomposition of the matrix (4) as

$$\mathbf{R} = \sum_{l=1}^{n}\lambda_l\mathbf{u}_l\mathbf{u}_l^H \qquad (7)$$

where λ_l, $l = 1, 2, \ldots, n$ are ordered (in a nonincreasing order) eigenvalues of $\mathbf{R}$, and $\mathbf{u}_l$ is the eigenvector corresponding to the lth eigenvalue λ_l. Taking (3) and (7) into account, we can rewrite (6) as

$$\tilde{\mathbf{R}} = \frac{1}{k}\sum_{i=1}^{k}\sum_{l=1}^{n}\lambda_l \mathrm{VP}(\mathbf{u}_l, i, m)\mathrm{VP}(\mathbf{u}_l, i, m)^H \qquad (8)$$

Therefore, the spatial smoothing preprocessing can be reformulated through the vector peeling process applied to the eigenvectors of $\mathbf{R}$. The signal subspace of the covariance matrix $\mathbf{R}$ in a coherent case is rank deficient and consists of the set of eigenvectors, corresponding to the nonzero eigenvalues of the matrix $\mathbf{ASA}^H$. In other words, if the rank of this matrix is p $(p \leq q)$, the signal subspace is given by the set of eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ corresponding to the p largest eigenvalues of $\mathbf{R}$. Thus, we have

$$\mathbf{R} - \sigma^2\mathbf{I}_n = \mathbf{ASA}^H = \sum_{l=1}^{p}(\lambda_l - \sigma^2)\mathbf{u}_l\mathbf{u}_l^H \qquad (9)$$

From (3) and (9) it follows that after the spatial smoothing preprocessing the $n \times n$ matrix $\mathbf{ASA}^H$ is transformed to the $m \times m$ matrix:

$$\widetilde{\mathbf{ASA}^H} = \frac{1}{k}\sum_{i=1}^{k}\sum_{l=1}^{p}(\lambda_l - \sigma^2)\mathrm{VP}(\mathbf{u}_l, i, m)\mathrm{VP}(\mathbf{u}_l, i, m)^H \quad (10)$$

As in spatial smoothing, the deficient signal subspace of dimension p is restored by vector peeling process. The peeled vectors are no longer the eigenvectors of the smoothed covariance matrix. From (10) it follows that the relationship between the vector sets corresponding to signal subspaces before and after spatial smoothing can be written as:

$$\{\mathbf{u}_l\}_{l=1,2,\ldots,p} \overset{\text{smoothing}}{\longrightarrow} \{\mathrm{VP}(\mathbf{u}_l, i, m)\}_{\substack{l=1,2,\ldots,p \\ i=1,2,\ldots,k}} \qquad (11)$$

Expression (11) represents the new way of restoring signal subspace through vector peeling which is theoretically similar but computationally different from spatial smoothing because latter is applied to the covariance matrix and the former to its eigenvectors. Since the dimension of signal subspace of the exact smoothed covariance matrix cannot exceed q, a subset of vectors in $\{\mathrm{VP}(\mathbf{u}_l, i, m)\}$ may be linearly dependent if $kp > q$. Furthermore, it follows from the equivalence in (8) that vector peeling requires the same number of minimum array elements to fully restore the dimension of the signal subspace as spatial smoothing and vice versa. In the finite number of snapshots (sample) case, (11) is also valid but vectors $\{\mathrm{VP}(\mathbf{u}_l, i, m)\}$ are linearly independent unless $kp > m$ because of the noisy character of covariance matrix eigenvectors. Nevertheless, both in the finite and infinite number of snapshots cases $kp - q$ vectors of the vector set $\{\mathrm{VP}(\mathbf{u}_l, i, m)\}$ are redundant. The situation in the sample case is complicated by the necessity to estimate the rank q of signal subspace in order to determine how much peeled vectors are redundant.

3. ALGORITHM DEVELOPMENT

In this section, we develop the coherent source location algorithm using vector peeling process. After a general formulation of the algorithm, we concretize each step taking into account both computational and performance factors. Then, we show that the computational load of our algorithm is less than that of spatial smoothing in the case of adaptive algorithms updating signal subspace dynamically [5].

Assume that the estimate $\hat{q}$ of the number of sources is available and that h $(h \leq p)$ eigenvectors $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \ldots, \hat{\mathbf{u}}_h$ corresponding to the largest eigenvalues of the array output sample covariance matrix are updated by one of the eigendecomposition algorithms [5]. The general form of the source location vector peeling algorithm can be written as the following sequence of steps:

- *Step 1:* Find the set of peeled vectors $\{\mathrm{VP}(\hat{\mathbf{u}}_l, i, m)\}$, $l = 1, 2, \ldots, h$, $i = 1, 2, \ldots, k$ as a result of applying the vector peeling process (1) to the vector set $\{\hat{\mathbf{u}}_l\}$, $l = 1, 2, \ldots, h$.
- *Step 2:* Choose $\hat{q}$ independent vectors $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{\hat{q}}$ from $\{\mathrm{VP}(\hat{\mathbf{u}}_l, i, m)\}$, $l = 1, 2, \ldots, h$, $i = 1, 2, \ldots, k$.
- *Step 3:* Calculate the MUSIC-type of DOA estimate:

$$f(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{P}_N\mathbf{a}(\theta)} = \frac{1}{m - \mathbf{a}^H(\theta)\mathbf{P}_S\mathbf{a}(\theta)} \qquad (12)$$

where the $m \times 1$ vector

$$\mathbf{a}(\theta) = (1, \exp\{-j\omega_0\tau\}, \ldots, \exp\{-j\omega_0(m-1)\tau\})^T$$

$\tau = (d/c)\sin\theta$, ω_0 is the center frequency, d is the interelement spacing, c is the propagation speed,

$$\mathbf{P}_S = \mathbf{G}(\mathbf{G}^H\mathbf{G})^{-1}\mathbf{G}^H, \quad \mathbf{P}_N = \mathbf{I} - \mathbf{P}_S \qquad (13)$$

is the $m \times m$ projection matrices onto the signal and noise subspaces, respectively, and

$$\mathbf{G} = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{\hat{q}}]$$

is a $m \times \hat{q}$ matrix.

Let us now concretize the steps 2 and 3 of this algorithm. Note that the vectors for constructing the projection matrix in (13) should be chosen from the peeled eigenvectors that correspond to the larger eigenvalues of the array output sample covariance matrix because they tend to be more stable [2]. If the total number of array sensors is large compared to the number of sources, then only one eigenvector (corresponding to the largest eigenvalue) may be enough for vector peeling (i.e., choose $h = 1$). Of course, the appropriate choice of m and k is necessary in this case. When an eigenvector $\hat{\mathbf{u}}$ is peeled, it is better to choose relatively well "separated" peeled vectors, i.e., vectors $\mathrm{VP}(\hat{\mathbf{u}}, i_1, m)$ and $\mathrm{VP}(\hat{\mathbf{u}}, i_2, m)$ such that $|i_1 - i_2|$ is large.

The main question that appears when implementing the algorithm is how to estimate the dimension of signal subspace. Possible solution to this problem is to use one of the existing coherent MDL criteria (see [6] and references therein). In this case, the estimate of $\hat{q}$ can be updated in parallel with signal subspace eigenvectors. Because of relatively high computational cost of the coherent MDL methods, it is convenient to update number of sources less often (i.e., with much smaller rate) than the eigenvectors. This means that employing the coherent MDL method, one should take much larger time intervals between neighboring updates of the number of sources than between the neighboring updates of highest eigenvectors. Such type of double-rate parallel updating of number of sources and eigenvectors is well motivated for the situations with moving sources, where continious (unceasing) changes of eigenvectors occur always while abrupt changes of number of sources occur from time to time only.

It is ideal from a computational point of view to take the orthonormalized vectors from the output of any vector orthonormalization procedure as the required vectors $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{\hat{q}}$. In this case, the equation (13) for the projection matrix is simplified to:

$$\mathbf{P}_S = \sum_{l=1}^{\hat{q}} \mathbf{g}_l \mathbf{g}_l^H \tag{14}$$

From these considerations, we can rewrite the algorithm in a more concrete and computationally effective form. Let us assume that only the first sample eigenvector $\hat{\mathbf{u}}_1$ is calculated. Then, the sequence of steps of vector peeling algorithm is as follows:

- *Step 1:* Find the set of peeled vectors $\{\mathrm{VP}(\hat{\mathbf{u}}_1, i, m)\}$, $i = 1, 2, \ldots, k$ using (1).
- *Step 2:* Apply one of the existing vector orthonormalization procedures [7] to the $\hat{q}$ vectors from this set. Involve in this procedure maximally "separated" peeled vectors (i.e., maximize the difference in index i between these vectors). On the output of this procedure, one have the orthonormal basis $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{\hat{q}}$ for signal subspace.
- *Step 3:* Calculate the MUSIC-type of DOA estimate (12) using (14).

On the Step 2, a variety of stable and computationally efficient vector orthonormalization procedures with complexity $O(q^2 m)$ are available [7].

4. COMPARISON OF COMPUTATIONAL LOADS

Let us now compare the computational loads of our algorithm in terms of complex flops[1] with that of spatial smoothing when applied to the DOA estimation problem using adaptive algorithms that update the signal subspace eigenvectors. The computational complexity of the fastest adaptive algorithms is $O(hr)$ flops per updating step [5] where h is the number of desired eigencomponents, and r is the eigenvector dimension. Here, we do not consider the computational loads of the MUSIC function calculation because they are the same for both algorithms. Therefore, spatial smoothing based technique requires $O(km^2)$ flops for covariance matrix smoothing and $O(qm)$ flops for updating algorithm itself, per step, respectively. Taking into account that $km \geq q$ always, we have that the complexity of spatial smoothing is $O(km^2) = O((n-m)m^2)$ per updating step. In turn, eigenvector peeling based technique requires $O(n)$ flops for updating the highest eigenvector and $O(q^2 m)$ flops for vector orthonormalization per updating step, respectively. Therefore the total complexity of eigenvector peeling is $O(q^2 m) + O(n)$ per step. Evidently, eigenvector peeling provides significant computational improvement.

For example, for large array and relatively small number of sources, the number of subarrays is typically chosen such that $k \gg q$ in order to provide the suitable compromise between the size of working aperture and the source decorrelation effect [8]. In this case, the computational complexity of spatial smoothing may be dramatically higher than that of vector peeling. For instance, for $n = 100$, $m = 60$, $q = 4$, we have that spatial smoothing requires $\sim 10^5$ flops, while eigenvector peeling requires only $\sim 10^3$ flops.

In addition to significant computational improvement, vector peeling provides more flexibility than spatial smoothing. Indeed, in spatial smoothing once the subarray dimension is chosen it is difficult to change it because this requires eigendecomposition for a matrix of different dimension. In vector peeling, the subarray dimension can be changed dynamically based on the environment (without reperforming the eigendecomposition).

[1]Each floating point operation (flop) is defined as either complex addition or complex multiplication [7].

5. SIMULATION RESULTS

In simulations, we assumed equispaced linear array of 10 omnidirectional sensors with $\lambda/2$ interelement spacing and 2 fully coherent sources impinging from $0°$ and $4°$ and having the phase difference $\pi/2$ in the first array sensor. The number of snapshots in each simulation run was equal to 100. In spatial smoothing, we assume $m = 7$ that corresponds to the optimal subarray choice [8]. In vector peeling, we assumed the same dimension of peeled vectors and use maximally "separated" vectors $\mathrm{VP}(\hat{\mathbf{u}}_1, 1, 7)$ and $\mathrm{VP}(\hat{\mathbf{u}}_1, 4, 7)$ for estimating the signal subspace.

It is well known that conventional MUSIC without any preprocessing fails in the coherent scenarios. We do not illustrate this fact because it is well documented in [3]. Fig. 1 shows ten plots of MUSIC spectra with spatial smoothing, while Fig. 2 shows ten plots of spectra (12) with vector peeling, respectively. In this example, SNR = 30 dB. From these figures, it is clear that both techniques have very similar spectral functions. Figs. 3 and 4 show the root-mean-square error (RMSE) and absolute bias of DOA estimation versus SNR, respectively, while Fig. 5 demonstrates the probability of resolution versus SNR. A total of 100 independent runs were performed to obtain each simulated point. The sources were considered as resolved in each run when DOA estimation error for each source was less than half of source separation.

The results of Figs. 3-5 demonstrate that vector peeling algorithm slightly outperforms spatial smoothing technique because the former has the lower SNR threshold. This establishes the feasibility of vector peeling as an alternative approach to spatial smoothing.

6. CONCLUSIONS

We propose a novel preprocessing technique which forms a basis for alternative *computationally efficient* solution of the coherent source location problem. It is shown that this technique is related to the well known spatial smoothing algorithm. We demonstrate that vector peeling slightly outperforms spatial smoothing in terms of RMSE, bias, and probability of source resolution when both these techniques are employed with the MUSIC-type DOA estimators. At the same time, vector peeling approach provides more computational advantages and flexibility than spatial smoothing.

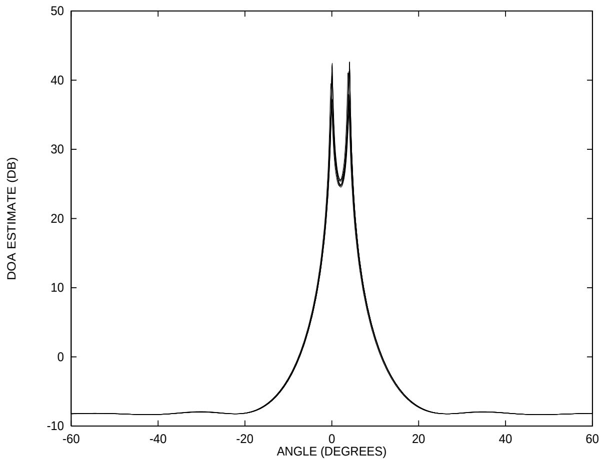

Figure 1: Ten MUSIC plots after spatial smoothing.

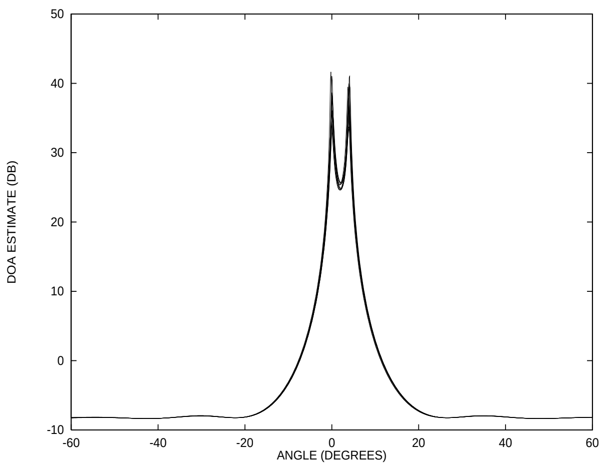

Figure 2: Ten plots (12) after vector peeling.

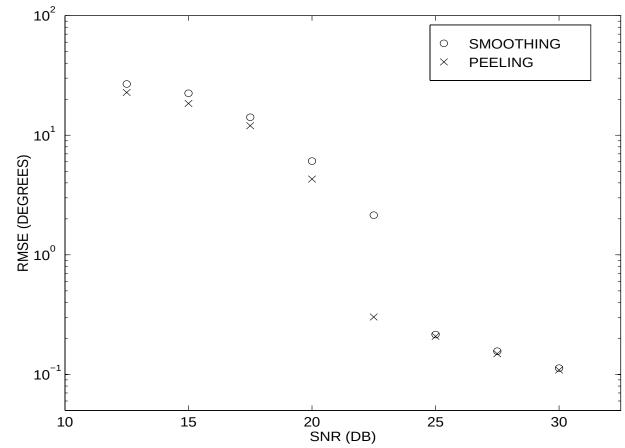

Figure 3: DOA estimation RMSE.

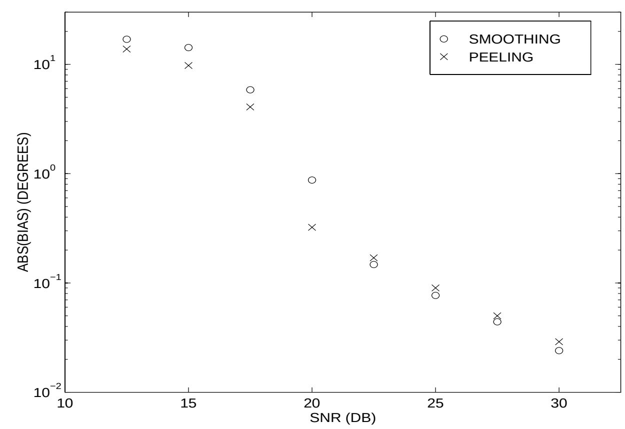

Figure 4: DOA estimation absolute bias.

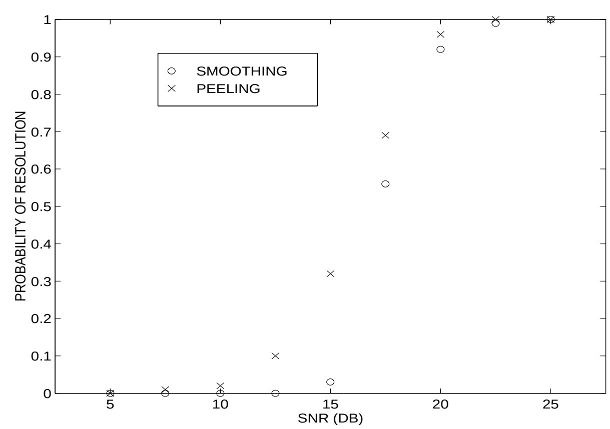

Figure 5: Probability of source resolution.

REFERENCES

[1] R.O. Schmidt, "Multiple emitter location and signal parameter estimation," in *Proc. RADC Spectral Estimation Workshop*, 1979, pp. 243–258.

[2] S.S. Reddi, "Multiple source location – a digital approach," *IEEE Trans. on AES*, vol. AES–15, pp. 95–105, 1979.

[3] T.J. Shan, M. Wax, and T. Kailath, "On spatial smoothing for direction-of-arrival estimation of coherent signals," *IEEE Trans. on ASSP*, vol. ASSP–33, pp. 806–811, Aug. 1985.

[4] J. Cardzow, "A high-resolution direction-of-arrival algorithm for narrow-band coherent and incoherent sources," *IEEE Trans. on ASSP*, vol. ASSP–36, pp. 965–979, July 1988.

[5] B. Yang, "Projection approximation subspace tracking," *IEEE Trans. on SP*, vol. SP–43, pp. 95–107, Jan. 1995.

[6] C.W. Ma, and C.C. Teng "Detection of coherent signals using weighted subspace smoothing," *IEEE Trans. on AP*, vol. AP–44, pp. 179–187, Feb. 1996.

[7] G.H. Golub, and C.F. Van Loan, *Matrix Computations*, second edition, the Johns Hopkins University Press, Baltimore, 1989.

[8] A.B. Gershman, and V.T. Ermolaev, "Optimal subarray size for spatial smoothing," *IEEE Signal Processing Letters*, vol. 2, pp. 28–30, Febr. 1995.