Convergence Analysis of a Variable Step-Size Normalized LMS Adaptive Filter Algorithm

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Abstract

This paper investigates the convergence properties of a variable step normalized LMS (VSNLMS) adaptive filter algorithm. Instead of a fixed step-size used in the conventional normalized LMS algorithm, the step-size of the algorithm under study is updated in each iteration, based on an expression related to the output errors. The variable step-size improves the convergence speed, while sacrificing little in complexity. For an application where the adaptive filter is used to track a time varying channel it is shown that the step-size converges towards its optimum value. Simulation results are presented to support the analysis.

I. Introduction

In recent years, many new adaptive systems with the capacity of tracking both stationary and nonstationary signals have been applied to practical use, such as echo cancellers in a conference room, or automatic equalizers in digital cellular radio systems which exhibit rapid fading problems. In such applications, a fast adaptive filter algorithm is necessary to cope with both stationary and nonstationary environments.

For complexity reasons, there is still strong interests on the study of simple and efficient LMS algorithms and of normalized LMS algorithms under various nonstationary conditions. It is well known that the mean square error of the LMS algorithm with nonstationary signals consists of the gradient noise and weight vector lag. Both components are functions of the step-size (μ). This makes the selection of the optimal step-size difficult. This fact has severely limited the usefulness of algorithms with fixed step-sizes in some applications. Moreover, the normalized LMS algorithm becomes unstable when the norm of the input vector tends towards zero. The problem is caused by the division in the procedure of the normalized LMS algorithm. A solution to this problem is to interrupt the coefficient update when the norm of an input vector is smaller than a threshold [2].

In reference [1], a variable step size LMS (VSLMS) algorithm is proposed which improves performance over the fixed step size LMS: at the beginning of the adaptation, the error is large, causing the step size to increase and provide faster convergence speed. When the error decreases, the step decreases thus yielding smaller misadjustment. Unfortunately, this algorithm is dependent on a priori knowledge about the statistics of the environment. In reference [2], a modified NLMS algorithm focused on the stability and convergence speed is presented, but its fixed step size limits its convergence speed.

These problems can be solved by the proposed approach, namely, a variable step-size normalized LMS adaptive algorithm which selects the optimal step-size interactively, and overcomes many of the limitations of the methods discussed above. This paper has two contributions, the first is the analysis the proposed algorithm. As the second, we derive the expressions for the mean square error and misadjustments of the filter, which give guidelines for the design of the adaptive filter. The results derived from the analysis are verified numerically through computer simulations for an example of adaptive equalization system.

2. The variable step size normalized adaptation.

The proposed adaptation consists of introducing the variable step size $\mu(n)$, updated by the expression

$$\mu(n) = \mu(n-1) + \rho e(n) e(n-1) X^T(n-1) X(n) \quad (1)$$

into the classical normalized LMS algorithm:

$$W(n+1) = W(n) + \frac{\mu(n)}{\|X(n)\|^2} X(n) e(n) \quad (2)$$

Here, X^T denotes the matrix transpose of *X*, *W(n)* is the coefficient vector of the adaptive filter at time n, which is defined by

$$W(n) = [w(1),\ w(2),\ \ldots,\ w(N)] \quad (3)$$

where N is the number of coefficients. X(n) is an input vector

$$X(n) = [x(n),\ x(n-1),\ \ldots,\ x(n-N+1)] \quad (4)$$

The output signal of the adaptive filter is expressed as

$$y(n) = W^T(n) X(n) \quad (5)$$

Let

$$d(n) = X^T(n) W_{opt}(n) + \xi(n) \quad (6)$$

where $\xi(n)$ corresponds to the optimal estimation error process; and

$$\xi_{\min} = E[\xi^2] \quad (7)$$

is the minimum value of the MSE.

Let $W_{opt}(n)$ denote the optimal coefficient vector for estimating the desired response signal *d(n)* using *X(n).* We assume that $W_{opt}(n)$ is time varying, and the time variations are caused by a random disturbance of the optimal coefficient process. Thus, the behavior of the optimal coefficient process can be modeled as

$$W_{opt}(n) = W_{opt}(n-1) + \eta(n-1) \quad (8)$$

where $\eta(n-1)$ is a zero-mean independent disturbance process with covariance matrix $\sigma_n^2 I$. We will assume that the triplet $\{X(n),\ \xi(n),\ \eta(n-1)\}$ is a statistically independent random process.

In Eq.(1), ρ is a small positive constant that controls the adaptive behavior of the step-size sequence $\mu(n)$. To assure convergence of the mean square error, it can be shown that a sufficient condition is [4].

$$0 < \mu(n) < \frac{2}{(3 + \frac{1}{M_{adj}}) tr(R)} \quad (9)$$

Where M_{adj} is the misadjustment level for the fastest convergence, defined by

$$M_{\text{adj}} = \frac{E_{\text{ex}}}{\zeta_{\min}} \quad (10)$$

It is known that the value of $\mu(n)$ is determined from Eq.(9) according to the final misadjustment requirement E_{ex}, so in the next section, we will derive the final E_{ex}.

It is noted that if $\mu(n)$ falls outside the range in (9), we can bring it inside the range by setting it to the closest of the boundaries of Eq. (9).

3. Convergence Analysis of VSNLMS

When operating in stationary and nonstationary noise environments, we consider the effect of the interruption of the coefficients update when the norm of input vector is smaller than the threshold:

$$E\left[\|X(n)\|^2\right] = \varepsilon \quad (11)$$

Let

$$V(n) = W(n) - W_{opt}(n) \quad (12)$$

denote the coefficient misadjustment vector at time n. The output error of the system is

$$e(n) = d(n) - X^T(n) W(n) \quad (13)$$

Substituting Eqs.(7), (8) and (3) in (12) results in

$$V(n+1) = \left[I - \frac{\mu(n)}{\|X(n)\|^2} X(n) X(n)^T \right] V(n) + \frac{\mu(n)}{\|X(n)\|^2} X(n) \xi(n) - \eta(n) \quad (14)$$

Now, using the independence assumption and the uncorrelation of $\mu(n)$ with *X(n)* and $\xi(n)$ and combining usual analysis techniques for Gaussian input with the approximation that $\mu(n)$ and $\mu^2(n)$ are uncorellated with the data,

The following equation for the second moment matrix of the coefficient vector is obtained

$$G(n+1) = E\left[V(n+1)V(n+1)^T\right] \tag{15}$$

and

$$G(n+1) = G(n) - 2E\left[\frac{u(n)}{\|X(n)\|^2}\right]\sigma_x^2 G(n) + E\left[\frac{u(n)^2}{\|X(n)\|^4}\right]$$

$$\left(2\sigma_X^4 G(n) + \sigma_x^2 \sigma_e^2(n) I\right) + \sigma_n^2 I \tag{16}$$

where

$$\sigma_e^2 = \xi_{\min} + \sigma_x^2 tr\left[G(n)\right] \tag{17}$$

The convergence value is defined using Eq.(16) as

$$E[\|G(\infty)\|^2] = \lim_{n\to\infty} E[V(n+1)V^T(n+1)] \tag{18}$$

and the mean and mean squared behavior of the step-size sequence $\mu(n)$ can be shown to follow the non-linear difference equations:

$$E[\mu(n)] = E[\mu(n-1)] + \rho \mathrm{E}\left[\mathrm{e(n)e(n-1)X^T}(n)X(n)\right] \tag{19}$$

In order to obtain the steady-state values $E[\mu(\infty)]$, and $E[\mu^2(\infty)]$ of the step-size, we have made use of the independence assumption, the uncorrelatedness of $\mu(n)$ with other quantities involved, and the fourth-order expression of Gaussian variables expressed as a sum of products of second-order expectations [3]. We will set $\sigma_n^2 = 0$ for stationary environments, and, after some simplifications, we get the convergence-value

$$G(\infty) \approx \left(0.5\xi_{\min}^3 \rho / N\sigma_x^2\right)^{\frac{1}{2}} \tag{20}$$

The corresponding steady-state excess-mean-squared estimation error is given by

$$E_{ex} = N\sigma_X^2 G(\infty) \approx \left(0.5N\sigma_x^2 \xi_{\min}^3 \rho\right)^{\frac{1}{2}} \tag{21}$$

ρ is usually chosen to be very small and this implies that E_{ex} is also very small. Now,

$$M_{adj} \approx (0.5N\sigma_x^2 \xi_{\min} \rho)^{\frac{1}{2}} \tag{22}$$

For nonstationary environments, $\sigma_n^2 \neq 0$, we get

$$E_{ex}(\infty) \approx \left(0.5N\sigma_x^2 \xi_{in}^3 \rho + \sigma_n^2 \sigma_x^2 \xi_{\min}\right)^{\frac{1}{2}} \tag{23}$$

hence

$$M_{adj} \approx (0.5N\sigma_x^2 \xi_{\min} \rho + \sigma_n^2 \sigma_x^2 / \xi_{\min})^{\frac{1}{2}} \tag{24}$$

From Eqs.(22), (24), in stationary and nonstationary applications, it follows that when ρ is chosen close to 0, the VSNLMS algorithm brings down the misadjustment to smaller values. That means that it is possible to get arbitrarily close to the optimal performance of VSNLMS by choosing ρ appropriately.

4. Simulation

The simulation results presented in this section will show the potential performance of the VSNLMS algorithm. Results will be presented for the adaptive equalization of a simple channel with various eigenvalue spreads and a time varying channel. This channel has been used already as a test case for various algorithms. The impulse response is

$$\mathrm{h(n)} = \begin{cases} \frac{1}{2}\left(1+\cos\left(\frac{2\pi(\mathrm{n}-2)}{\mathrm{W}}\right)\right) & \mathrm{n} = 1,2,3 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

Where the eigenvalue spread is 6.08 for *W*=2.9 and 21.71 for *W*=3.3. The adaptive equalizers have 11 taps. The simulations include an additive noise power of $\sigma_n^2 = 0.001$, which gives $\xi_{\min}$ = 0.003178 for *W*=2.9, and $\xi_{\min}$ =0.002476 for *W*=3.3.

Figure 1 shows that the speed of convergence of the VSNLMS algorithm is essentially unaffected by the eigenvalue spread. However, the computer simulations show that the step-sizes of the VSNLMS algorithm do not convergence towards their optimum values when the input samples to the adaptive filter are highly correlated, so that the ratio of the maximum to minimum eigenvalues of R is in the order of hundreds or above. This algorithm converges much faster than the VSLMS, and the steady-state value of the averaged squared error produced by the algorithm is much smaller than in the case of the LMS algorithm. In Fig.2, we observe that the VSNLMS algorithm is able to track an abruptly changing channel. Results are obtained by averaging over 200 independent runs (ρ =0.0005).

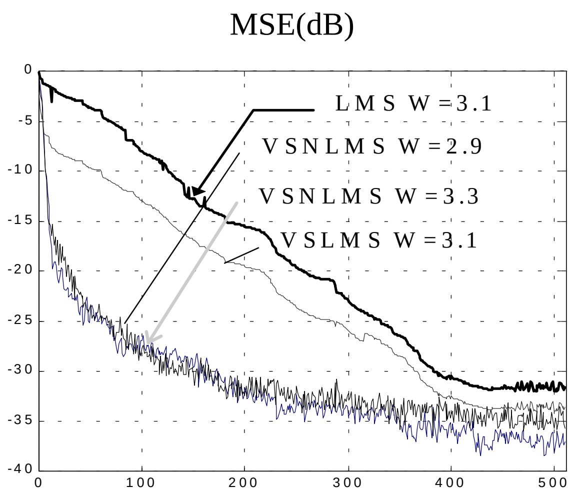

Fig.1 Simulation for the LMS, VSLMS and VSNLMS algorithms for channel with different eigenvalues spread ($\mu(0) = 0.005$) .

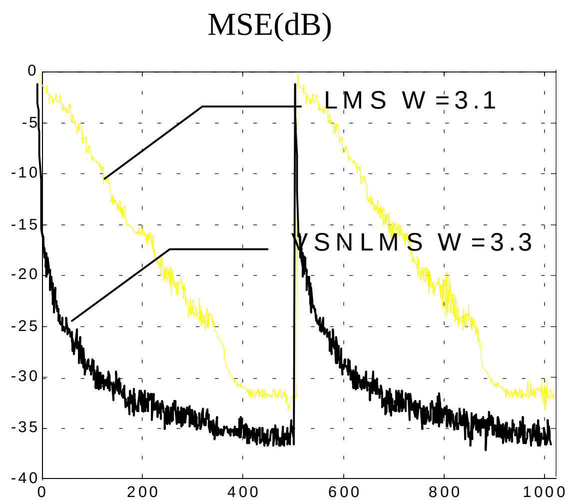

Fig.2 Simulation for the VSNLMS filter algorithm for an abruptly changing channel.

5. Conclusions

In this paper, we have studied a variable step-size normalized LMS adaptive filtering algorithm, which overcomes the disadvantages of the selection of step-size μ and misadjustement of LMS algorithm. To counter the unstable behavior of the VSNLMS algorithm when all the elements of an input vector are very small at the same time, we use a threshold to interrupt the adaptive filter coefficient update. The simulation results show that the initial convergence speed has a significant convergence rate improvement over LMS. The algorithm seeks to adjust the step-size in the direction of the optimal value in the case of nonstationary environments, and the algorithm is relatively insensitive to the eigenvalue spread of the input data.

Reference

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