PERFORMANCE OF AN OPTIMAL MULTIPLICATIVE JUMP DETECTOR BASED ON THE CONTINUOUS WAVELET TRANSFORM

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ABSTRACT

Additive and multiplicative abrupt changes in random signals have been studied in many applications. In segmentation theory, the detection of these additive abrupt changes allows the determination of stationary parts of signals. In radar images, multiplicative abrupt jumps have been used to model "speckled" signal: these multiplicative jumps correspond to object edges on piecewise constant backgrounds. The Continuous Wavelet Transform (CWT) has shown nice properties for the detection of abrupt additive jumps. The paper studies the problem of abrupt multiplicative jump detection using the CWT. The time-scale plane Neyman-Pearson test is studied and its performance is evaluated.

1 INTRODUCTION

The problem of additive abrupt jumps has been long considered. Additive jumps are used in line-by-line edge detection on digitalized images or in bar code modelling [1]. However, in many applications, multiplicative abrupt changes more accurately model the signals. These applications include speckle signal in radar images [2], mechanical vibrations, non-linear time series and random communication models [1]. The Continuous Wavelet Transform (CWT) has been shown to be an effective tool for the detection of additive jumps [3]. Additive and multiplicative jumps have proportional signatures in the time-scale plane [4]. This property has motivated the use of the Continuous Wavelet Transform for the detection of multiplicative jumps. The Neyman-Pearson (NP) test is developed in the (time, scale) plane for the multiplicative case. The NP test is optimal in the sense that it maximizes the probability of detection (pd) for fixed probability of false alarm (pfa). It provides a reference to compare with suboptimal detectors. The performance of the NP test is studied in terms of pfa and pnd (probability of non detection). The Receiver Operating Characteristics (ROC) curves display the test performance as a function of the jump amplitude.

2 DECISION PROBLEM

2.1 Binary Hypotheses Test

A shifted step multiplied by a random Gaussian non-zero mean process is considered, for simplicity. This model leads to simultaneous mean value and variance jumps. If the signal parameters are known a priori, the optimum detector reduces to a simple binary hypothesis test. The two hypotheses can be expressed as follows:

- Under hypothesis H_0, the observed process $y(t)$ is a stationary non-zero mean Gaussian process $x(t)$:

$$y(t) = x(t) \qquad m_x \neq 0 \tag{1}$$

- Under hypothesis H_1, the process $x(t)$ is multiplied by a shifted step of amplitude A at time t_0 :

$$y(t) = x(t)s(t) = x(t)\left[1 + AU(t - t_0)\right] \tag{2}$$

2.2 Test formulation in the time-scale plane

According to [4], the CWT is an effective tool for the detection of abrupt multiplicative jumps. The CWT is defined by:

$$C_y(a,\tau) = \frac{1}{\sqrt{a}} \int_R y(t)\psi^*\left(\frac{t-\tau}{a}\right) dt \tag{3}$$

$\psi(\mathrm{t})$ is an L^2 normalized wavelet with bounded support $\left[-\frac{\Delta t}{2}, \frac{\Delta t}{2}\right]$. The symmetric Haar wavelet is used since it has shown interesting properties for the detection of abrupt multiplicative jumps [4].

The CWT is sampled at scales $a_i \quad i \in \{1, 2, ..., p\}$ and times $\tau_j \quad j \in \{1, 2, ..., m\}$. Denote

$$C_y(a_i) = \left[C_y(a_i, \tau_1)\ C_y(a_i, \tau_2)\ ...\ C_y(a_i, \tau_m)\right]^t \tag{4}$$

as the wavelet transform vector of the process $y(t)$ at the fixed scale a_i. Denote

$$C_y = \left[C_y(a_1)^t\ C_y(a_2)^t\ ...\ C_y(a_p)^t\right] \tag{5}$$

as the n-dimensional wavelet transform vector with $n = m \times p$. Due to the linearity of the transform, C_y is Gaussian under both hypotheses.

- Under hypothesis H_0, the CWT vector Cy is a n-dimensional Gaussian process with mean vector $M_0 = 0$ and covariance matrix denoted by μ_0.
- Under hypothesis H_1, the CWT vector Cy is a n-dimensional Gaussian process with mean vector $M_1 \neq 0$ and covariance matrix μ_1. The mean vector corresponds to the conic jump signature due to the mean value jump [4].

Because of the variance jump, the covariance matrices under the two hypotheses are different. This is not the case for additive jumps.

2.3 Neyman-Pearson Test

The Neyman-Pearson test for the Gaussian n-dimensional case can be applied on the CWT vector Cy. The likelihood ratio $\Lambda(C_y)$ is compared to a threshold k depending on pfa :

$$H_1 \text{ is rejected if } \quad \frac{L\left(C_y \mid H_0\right)}{L\left(C_y \mid H_1\right)} > k(pfa) \tag{6}$$

The sufficient statistics [5] expressed as:

$$l(C_y) = C_y^t\left(\mu_1^{-1} - \mu_0^{-1}\right) C_y - 2M_1^t \mu_1^{-1} C_y \tag{7}$$

leads to an equivalent test:

$$H_1 \text{ is rejected if } \qquad l\left(C_y\right) > d \tag{8}$$

with:

$$d = 2\log(k(pfa)) + \log\left[\frac{\det \mu_0}{\det \mu_1}\right] - M_1^t \mu_1^{-1} M_1 \tag{9}$$

Note that the optimal multiplicative jump detector is non linear whereas it is linear for additive jumps [3]. This complicates the computation of the test performance since the test statistic is non-Gaussian.

3 TEST STATISTIC

3.1 Test Statistic Distribution

The probability density function (p.d.f.) of $l(C_y)$ under H_0 and H_1 has to be determined in order to evaluate the test performance. Introducing the unit normal n-dimensional variable $W = [w_1\ w_2\ ...\ w_n]$:

$$\begin{aligned} W &= \mu_0^{-1/2} C_y \text{ under } H_0 & (10)\\ W &= \mu_1^{-1/2}(C_y - M_1) \text{ under } H_1 & (11)\end{aligned}$$

$l(C_y)$ can be expressed as two different quadratic forms of W (Q_0 under H_0 and Q_1 under H_1) plus an additive constant which is independent of the hypotheses:

$$\begin{aligned} Q_0(W) &= (W - F_0)^t E_0 (W - F_0)\\ Q_1(W) &= (W - F_1)^t E_1 (W - F_1)\end{aligned} \tag{12}$$

with:

$$\begin{aligned} E_1 &= \mu_1^{\frac{1}{2}}\left(\mu_1^{-1} - \mu_0^{-1}\right)\mu_1^{\frac{1}{2}} & (13)\\ E_0 &= \mu_0^{\frac{1}{2}}\left(\mu_1^{-1} - \mu_0^{-1}\right)\mu_0^{\frac{1}{2}} & (14)\\ F_1 &= \mu_1^{-\frac{1}{2}}\left(\mu_1^{-1} - \mu_0^{-1}\right)\mu_0^{-1} M_1 & (15)\\ F_0 &= \mu_0^{-\frac{1}{2}}\left(\mu_1^{-1} - \mu_0^{-1}\right)\mu_1^{-1} M_1 & (16)\end{aligned}$$

E_0 and E_1 are symmetric negative matrices. Indeed, in the case of a white Gaussian noise:

$$\mu_1 = \mu_0 + \left(2A + A^2\right) cov\left(C_{x(t)U(t-t_0)}\right) \tag{17}$$

where $cov\left(C_{x(t)U(t-t_0)}\right)$ is the covariance matrix of $x(t)U(t-t_0)$ CWT vector. Equation (17) leads to:

$$\mu_1^{-1} - \mu_0^{-1} = -\left(2A + A^2\right)\mu_1^{-1} cov\left(C_{x(t)U(t-t_0)}\right)\mu_0^{-1} \tag{18}$$

Hence:

$$\begin{aligned} E_0 &= -\left(2A + A^2\right)\mu_0^{\frac{1}{2}}\mu_1^{-1} cov\left(C_{x(t)U(t-t_0)}\right)\mu_1^{-\frac{1}{2}} & (19)\\ E_1 &= -\left(2A + A^2\right)\mu_1^{-\frac{1}{2}} cov\left(C_{x(t)U(t-t_0)}\right)\mu_0^{-1}\mu_1^{\frac{1}{2}} & (20)\end{aligned}$$

Eq. (19) and (20) show that matrices -E_0 and -E_1 are the product of symmetric positive matrices. Consequently, E_0 and E_1 are symmetric negative and can be diagonalized as:

$$\begin{aligned} E_1 &= V_1 D_1 V_1^t & (21)\\ E_0 &= V_0 D_0 V_0^t & (22)\end{aligned}$$

V_0 and V_1 are the eigenvector matrices of E_1 and E_0. D_0 and D_1 are the eigenvalue matrices of E_1 and E_0. $Q_0(W)$ and $Q_1(W)$ can be expressed as:

$$\begin{aligned} Q_0(W) &= (W - F_0)^t V_0 D_0 V_0^t (W - F_0)\\ &= Z_0^t D_0 Z_0 & (23)\\ Z_0 &= V_0^t W - V_0^t F_0 = G - F_0' & (24)\end{aligned}$$

$$\begin{aligned} Q_1(W) &= (W - F_1)^t V_1 D_1 V_1^t (W - F_1)\\ &= Z_0^t D_0 Z_0 & (25)\\ Z_1 &= V_1^t W - V_1^t F_1 = G - F_1' & (26)\end{aligned}$$

where G is the unit normal n-dimensional vectors. Eq. (23) and (25) lead to:

$$\begin{aligned} Q_0(W) &= \sum_{j=1}^{n} \lambda_{0j}\left(g_{0j} - f_{0j}'\right)^2 & (27)\\ 0 &\geq \lambda_{01} \geq \lambda_{02} \geq ... \geq \lambda_{0n} \end{aligned}$$

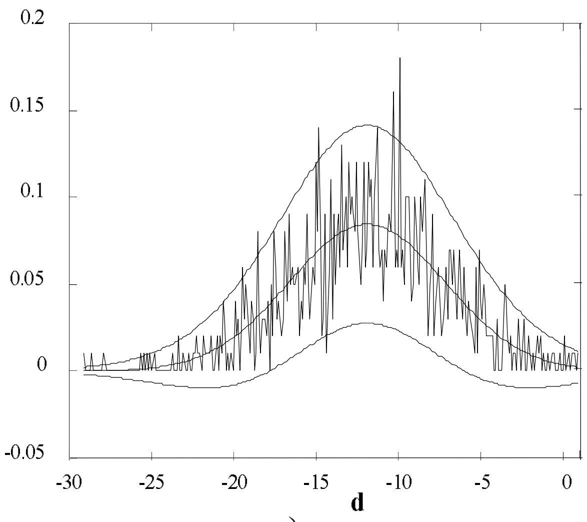
$$\begin{aligned} Q_1(W) &= \sum_{j=1}^{n} \lambda_{1j}\left(g_{1j} - f_{1j}'\right)^2 & (28)\\ 0 &\geq \lambda_{11} \geq \lambda_{12} \geq ... \geq \lambda_{1n} \end{aligned}$$

λ_{0i} and λ_{1i} denote the eigenvalues of E_0 and E_1. Equations (27) and (28) show that $Q_0(W)$ and $Q_1(W)$ can be expressed as weighted sums of squared shifted one-dimensional unit Gaussian variables. Expressions of the cumulative distribution functions of $Q_0(W)$ and $Q_1(W)$ (and of the corresponding probability density functions) can then be derived. The cumulative distribution functions can be expanded in series of central χ^2, non-central χ^2 cumulative distribution functions as well as in Laguerre and Maclaurin series [6]. Consequently, the test statistic distribution under both hypotheses can be theoretically derived. The resulting expressions are difficult to handle which complicates the computation of the test performance. To cope with this problem, an approximation of the test statistic distribution is studied in the next section.

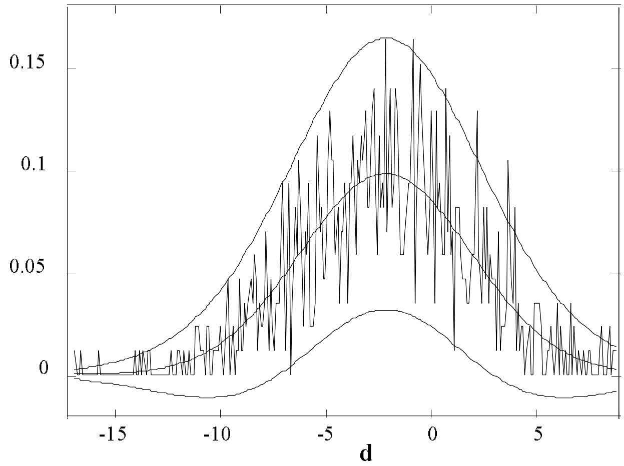
3.2 Approximation of the Test Statistic Distribution

According to (27) and (28) , Q_0 and Q_1 can be expressed as the sum of n variables whose distribution is a non central χ^2 distribution. Because of the dependance of these variables, the central limit theorem can not easily be applied. However, it seems reasonable to approximate the distribution of Q_0 and Q_1 by the Gaussian distribution, for large values of n.

As an example, consider $N = 2048$ samples of a Gaussian distributed random sequence with $m_x = 1$ and variance $\sigma_x^2 = 1$. The multiplicative jump is defined by $A = 0.1$ and $t_0 = 1024$. The CWT is computed at scales [10 20 30 40 50] and times [512 768 1024 1280 1536] with the symmetrical Haar wavelet of support 20. Fig.1. shows that the test statistics can be approximated by Gaussian distribution.



a)



b)

Fig. 1 : Histograms of $l\,(C_y)$ and Gaussian p.d.f. with 95% confidence intervals. a) under H_1, b) under H_0

4 NP TEST PERFORMANCE

Denote $p_{l|\mathrm{H}_0}$ and $p_{l|\mathrm{H}_1}$ as the probability density functions ($p.d.f.$) of the test statistic under H_0 and H_1. The probabilities of false alarm (pfa) and of non detection (pnd) can be expressed as:

$$\begin{aligned} pnd &= P\left[\text{choose } \mathrm{H}_0 \ \ |\mathrm{H}_1 \text{ true}\right] && (29) \\ &= \int_d^{+\infty} p_{l|\mathrm{H}_0}\left(l \mid H_0\right) dl && (30) \end{aligned}$$

$$\begin{aligned} pfa &= P\left[\text{choose } \mathrm{H}_1 \ \ |\mathrm{H}_0 \text{ true}\right] && (31) \\ &= \int_{+\infty}^{d} p_{l|\mathrm{H}_1}\left(l \mid H_1\right) dl && (32) \end{aligned}$$

Denote m_0, m_1, σ_0^2, σ_1^2 as the mean values and variances of $l\,(C_y)$ under hypotheses H_0 and H_1. Under the assumption that $l\,(C_y)$ is a Gaussian variable under both hypotheses, the following result can be obtained:

$$pnd = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{d - m_1}{\sqrt{2}\sigma_1}\right)\right] \tag{33}$$

$$pfa = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{d - m_0}{\sqrt{2}\sigma_0}\right)\right] \tag{34}$$

In (33) and (34), the erf function is defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{35}$$

Variations of pnd and pfa as functions of the threshold d are plotted in Fig. 2 for different values of the jump amplitude A. For large values of A, the NP test shows good performance.

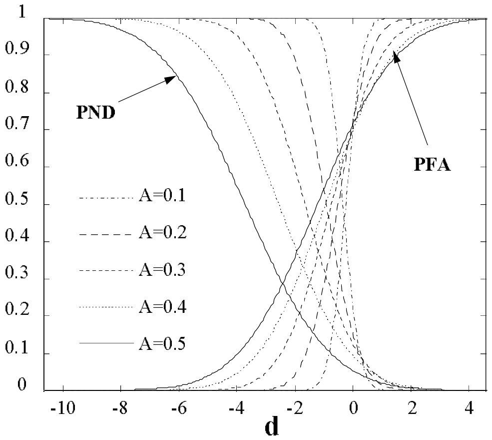


Fig. 2: Neyman-Pearson test performance as a function of the threshold d for different values of the jump amplitude

This representation allows to determine easily the threshold corresponding to fixed values of pnd and pfa. However, in order to compare with other detectors, a standard representation is proposed as well: a plot of the probability of detection , defined as

$$pd = P[\text{choose H}_1 \ |\text{H}_1 \text{ true}] \tag{36}$$
$$= \int_{-\infty}^{d} p_{l|\text{H}_0}(l \mid H_0)\, dl \tag{37}$$

versus pfa for different values of the jump amplitude is shown in Fig.3. The corresponding curves are usually referred to as Receiver Operating Characteristics (ROC) curves.

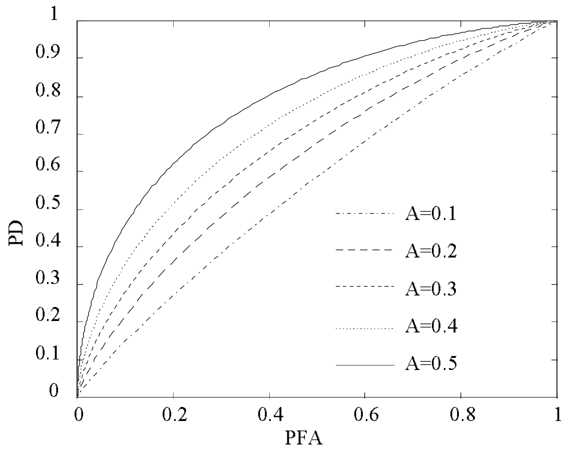


Fig.3: ROC curves for different jump amplitudes

The performance of the NP test in the time-scale plane has to be compared with the NP test for the signal vector $Y = [y(1), \ y(2), ..., \ y(N)]^t$ [7]. Under both hypotheses, Y is a Gaussian vector. This leads to a statistic distribution very similar to (7).

5 CONCLUSION

The optimal Neyman-Pearson multiplicative jump detector performance is derived in the time-scale plane. For fixed probability of false alarm, the threshold minimizing the probability of non detection can be determined. The performance for a given jump amplitude can be determined using ROC curves. This test constitutes a reference to which suboptimal detectors must be compared. In practical applications, the parameters of the jump and the multiplicative noise have to be estimated. These parameters can be estimated using the maximum likelihood estimator [7]. However, they can be estimated as well in the time-scale plane [8].

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