A SHORT WAY TO COMPUTE WT FROM STFT

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ABSTRACT

Wavelet transforms are often related to Fourier transforms due to their similitude and to underline some applications where wavelet theory should be superior over Fourier analysis. Many researchers have identified that the wavelet transform (WT) maps a function analogous to the short-time Fourier transform (STFT) that has a changing size. The increase numbers of FFT dedicated devices and of previous STFT databases request for a new approach between STFT and WT. This is one of the goals of the present paper. It also suggests a possible relation between the energetic representation of these transforms. The scalogram yields a graphical representation of the signal's energy distribute over the time-scale plane, as a spectrogram distributes the energy over the time-frequency plane. So, an interesting problem, the mapping between the spectrogram and the scalogram is derived. Few examples and some computational considerations are also provided.

NOTATIONS

In this paper any function is described by minor letters and its Fourier transform is written with capital letters. For a time-varying signal $x(t)$ the STFT and WT are computed with

$$F_x(\Omega,\tau) = \int_{-\infty}^{\infty} x(t) g^*(t-\tau) e^{-j\Omega t} dt \qquad (1)$$

$$W_x(a,b) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} \psi^*(\frac{t-b}{a}) x(t) dt \qquad (2)$$

where $a \in R^+$, $b \in R$, $g(t)$ and $\psi(t)$ are the window function and respectively the wavelet.

1 BRIEF REVIEW OF WT AND MALLAT'S ALGORITHM

The general description of time-frequency methods used the most fundamental one: the Wigner-Ville Distribution (WVD). Recently most authors have focused mainly in STFT and WT. The widely used Fourier transform is very convenient for describing the features of the signal. The information provided by this decomposition generates a complete description of stationary signals; however, the Fourier transform encounters serious restrictions when the characteristics of the signal to be analyzed are strongly time dependent. Therefore a satisfying description of non-stationary signals demands for other analysis methods. Hence, the spread of time-frequency (STFT) and time-scale analyses (WT).

Just in any analysis, two important problems appear in wavelet theory, one being the reverse of the other. First, being given a sequence of closed spaces, derived from a practical problem, which satisfies the multiresolution approximation condition[2], the problem is to find the scaling function $\phi(t)$ and then to compute the wavelet transform according (2). Mallat has proved that it can be derived an orthonormal wavelet basis from any multiresolution approximation [3]. Secondly, being given the scaling function the problem is to generate a multiresolution approximation. The solution is not always possible. Indeed, it is not true that we can built a multiresolution approximation from any wavelet orthonormal basis (see the counterexample given by Y.Meyer quoted on [2]). It might however be sufficient to impose a regularity condition on $\psi(t)$, in order to be possible always to generate a multiresolution approximation.

Now we are interested in another question that arises already. Being given a scaling function $\phi(t)$ how we compute the mother wavelet and then the wavelet transform? In some cases the problem was solved([1],[2],[3]). Let's look once again in [2] where the Mallat's Algorithm is described. The function $H(\Omega)$

$$H(\Omega) = \sum_{k=-\infty}^{\infty} h_k e^{-jk\Omega} \qquad (3)$$

must satisfy the conditions[2]:

$$\begin{gathered} |h_k| = O(1+k^2)^{-1} \\ |H(0)| = 1 \\ |H(\Omega)|^2 + |H(\Omega+\pi)|^2 = 1 \\ H(\Omega) \neq 0, \forall \Omega \in [-\pi/2, \pi/2] \end{gathered} \qquad (4)$$

The condition on $H(\Omega)$ are sufficiently in order to compute the Fourier transform of the corresponding function $\phi(t)$ with the relation:

$$\Phi(\Omega) = \prod_{k=1}^{\infty} H(2^{-k}\Omega) \quad (5)$$

Also it satisfies the following equation:

$$\Phi(2\Omega) = H(\Omega)\Phi(\Omega) \quad (6)$$

$H(\Omega)$ represents the high-pass filtering operator.

The low-pass filtering operator $G(\Omega)$ is given by the condition that:

$$\begin{bmatrix} H(\Omega) & G(\Omega) \\ H(\Omega+\pi) & G(\Omega+\pi) \end{bmatrix} \quad (7)$$

is a unitary matrix.

A possible choice for $G(\Omega)$ is given by (8):

$$G(\Omega) = e^{-j\Omega}\overline{H(\Omega+\pi)} \quad (8)$$

and it gives the Fourier transform for $\psi(t)$:

$$\Psi(2\Omega) = G(\Omega)\Phi(\Omega) \quad (9)$$

2 A NEW APPROACH BETWEEN STFT AND WT DERIVED FROM MALLAT'S ALGORITHM

Our aim is to find a direct relation between $\phi(t)$ and $\psi(t)$ without an intervention of $H(\Omega)$[8]. For that we solved the system obtained from (6)-(9) and we obtain the following relation:

$$\Psi(\Omega) = e^{-j\frac{\Omega}{2}}\overline{\Phi}(\Omega+2\pi)\cdot\overline{\Phi^{-1}}(\frac{\Omega}{2}+\pi)\Phi(\frac{\Omega}{2}) \quad (10)$$

We denote by:

$$\Gamma(\Omega) = \overline{\Phi}(\Omega)\overline{\Phi^{-1}}(\frac{\Omega}{2}); \gamma(t) = \mathcal{F}^{-1}[\Gamma(\Omega)] \quad (11)$$

then

$$\psi(t) = 0.5(e^{-j2\pi t}\gamma(t)) * \phi(2t-1) \quad (12)$$

Developing the convolution in the wavelet formula

$$\psi(t) = \frac{1}{2}e^{-j2\pi t}\int_{-\infty}^{\infty} e^{j2\pi\tau}\gamma(t-\tau)\phi(2\tau-1)d\tau \quad (13)$$

it holds the equation (14):

$$W_x(a,b) = \frac{e^{-j2\pi\frac{b}{a}}}{2\sqrt{a}}\int_{-\infty}^{\infty} e^{j2\pi\frac{t}{a}}P(a,b,\phi,t)x(t)dt \quad (14)$$

where we denote by $P(a,b,\phi,t)$ the integral

$$\int_{-\infty}^{\infty} e^{j2\pi\tau}\gamma^*(\frac{t-b}{a}-\tau)\phi(2\tau-1)d\tau \quad (15)$$

It is obviously that P is a Fourier transform which can be computed only from the scaling function before the processing of the input signal. The WT is reduced to compute a STFT with P as window function.

We interchange the order of integration in (14) and obtain the second approach

$$W_x(a,b) = \frac{e^{-j2\pi\frac{b}{a}}}{2\sqrt{a}}\int_{-\infty}^{\infty} e^{j2\pi\tau}Q(.,\tau)\phi(2\tau-1)d\tau \quad (16)$$

where we denote by $Q(.,\tau)$

$$\int_{-\infty}^{\infty} e^{j2\pi\frac{t}{a}}\gamma^*(\frac{t-b}{a}-\tau)x(t)dt \quad (17)$$

Similar considerations related to Fourier transforms as previously shown can be done for Q and second approach.

Let's now proceed with the well-known family of scaling functions: the set of cardinal B-splines [6]. The cardinal B-spline of order 1 is the box function $n_1(t) = \chi_{[0,1]}(t)$.

For $m > 1$ the cardinal B-spline $n_m(t)$ is defined recursively as a convolution

$$n_m = n_{m-1} * n_1 \quad (18)$$

These functions satisfy

$$n_m(t) = 2^{-1+m}\sum_{k=0}^{m}\binom{m}{k}n_m(2t-k) \quad (19)$$

$$N_m(\Omega) = (\frac{1-e^{-j\Omega}}{j\Omega})^m \quad (20)$$

In this case

$$\Gamma_m(\Omega) = 2^{-m}\sum_{k=0}^{m}\binom{m}{k}e^{jk\frac{\Omega}{2}} \quad (21)$$

$$\gamma_m(t) = 2^{-m}\sum_{k=0}^{m}\binom{m}{k}\delta(t+\frac{k}{2}) \quad (22)$$

$$Q_m = 2^{-m}a\sum_{k=0}^{m}\binom{m}{k}e^{j2\pi(\tau-\frac{k}{2}+\frac{b}{a})}x(a\tau+b-\frac{ak}{2}) \quad (23)$$

and finally

$$W_x(a,b) = \frac{\sqrt{a}}{2^{m+2}}\sum_{k=0}^{m}\binom{m}{k}(-1)^k \cdot \int_{-\infty}^{\infty} e^{j2\pi t}x(a\frac{t-k+1}{2}+b)n_m(t)dt \quad (24)$$

In some cases, it is probably that the last equation conducts to some extended computation time, but now it is clear we can use FFT dedicated devices for computing WT.

All these previous considerations are related to the continuous transforms. It is already well known that the complexity of the discrete wavelet transform is at most twice of its first stage. This results in a complexity of order $O(L)$ per input sample where L is the filter length.

Some correlations with the fast Fourier Transform (FFT) are often done. We seldom determine an FFT of the total signal. Instead we rather take a short-time Fourier transform (STFT) which has $O(logM)$ complexity per input sample (M being the number of frequency points, and the window length being a small multiple of M typically).

Multirate structures used for the discrete wavelet transform (DWT), are difficult to implement, because several signals have widely different sampling rates, requiring complex multiplexing and resource allocation. Since the DFT is based on convolution computation, it can be sped up by using the FFT to compute the various convolutions. This reduces the complexity from $O(L)$ to $O(logL)$.

It is clear now that the block processing structure of the STFT is favoured for implementations of WT.

3 MAPPING SPECTROGRAM TO SCALOGRAM

The Fourier spectrogram, defined as the square modulus of the STFT, is a very common tool in signal analysis because it provides a distribution of the energy of the signal in the time-frequency plane. A similar distribution can be defined in the wavelet case. This leads to wavelet scalogram, as the square modulus of the WT, which is a distribution of the energy of the signal in the time-scale plane. In contrast to the spectrogram, the energy of the signal is distributed with different resolutions in the case of spectrogram.

Since there are available a lot of spectrograms of test signals, we are interesting in a way to construct the scalogram for a given spectrogram and the reverse problem. Recent work [4] has shown that it is even possible to go continuously from the spectrogram of a given signal to its scalogram. More precisely, we start from the Wigner-Ville distribution, by progressively controlling Gaussian smoothing functions. Then we pass through a set of energy representations that either tends to the spectrogram if regular two-dimensional smoothing is used, or to the scalogram if "affine" smoothing is used. This property may also permit us to judge whether or not we should select time-scale analysis tools, rather time-frequency ones for a given issue.

The natural algorithm first introduced by us is based on resynthesize of the signal. Both the spectrogram and the scalogram produce a more or less easily interpretable visual two-dimensional representation of signals. However, such an energy representation has some disadvantages, too. In most of the cases the spectrogram, as well as the scalogram, cannot be inverted. Phase information is always necessary to reconstruct the signal. In the wavelet case the phase representation is more important. It has been shown that this shows narrow bursts in a signal than the scalogram does. For both transforms the successive steps are the choices of analysing functions (Gabor function and wavelet) and analysing grid, the computation of the transform, the choice of the reconstruct formula and finally the computation of the resynthesis. Applying first the synthesis of the signal with one transform and the analysis with the different one, we easy derived a procedure for mapping between spectrogram and scalogram [7]. The preceding approaches between WT and STFT recommend a potential connection between the energetic representation of these transforms, without the reconstruction of signal.

4 FINAL REMARKS AND CONCLUSIONS

Some considerations of the relation between the mother wavelet and the scaling function have already exposed. Under some conditions, a general form of the mother wavelet is retrieved. We are interested to derive a functorial approach between the transforms in digital signal processing, particularly for STFT and WT. This can leads to an easy implementation of the transforms and to a better understanding and relationships in DSP software.

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