

# ASYMPTOTIC PERFORMANCE ANALYSIS OF THE SINGLE-CYCLE DETECTOR

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## ABSTRACT

The paper deals with the analytical performance of the single-cycle detector, which is based on the cyclostationary properties of the signal to be intercepted. The Receiver Operating Characteristics (ROC) are derived theoretically, in discrete time, by using the asymptotic complex normality and covariance expressions of the sample average estimator of the cyclic-covariance when some “mixing conditions” are verified. Performance analysis of the single-cycle detector is evaluated for a cyclostationary signal observed in a background of stationary, zero-mean, white Gaussian noise. A numerical example for interception of a Binary-Phase-Shift-Keying (BPSK) signal is considered.

## 1 Introduction

Recently, cyclic detectors [2] have been proposed in interception problems, in order to exploit the spectral correlation property of the cyclostationary signal of interest (SOI) to be intercepted. This approach supposes knowledge of a few characteristics of the SOI, such as the modulation type (e.g., hop rate, keying rate, etc.). The optimum detector (Neyman-Pearson sense), in a “weak signal” assumption, is the multi-cycle (MC) detector, which is derived from the Locally Optimum (LO) test and the cyclostationary SOI model  $s(t)$ .

However, appropriate implementation of the MC detector requires knowledge of the SOI phases (e.g. carrier phase and keying clock phase for PSK signals). Hence, in order to avoid the problem of unknown phases, a sub-optimum structure is used, referred to as the single-cycle (SC) detector, employing only one cyclic frequency and taking the magnitude of the statistic [2],[3]. In each of these works the performance of the SC detector is characterized using the analytic measure of deflection (by omitting the magnitude operation for tractability) or using ROC obtained via simulation.

The purpose of this paper is to approach the exact evaluation of the detection probability and of the false alarm probability.

Conditional probability density function (pdf) of the SC detector under null and alternative hypotheses is de-

rived by using results based on the asymptotic complex normality (ACN) of the cyclic correlogram, which is a consistent estimator of the cyclic covariance when some “mixing conditions” are verified [1]. Analytic expression of the performance of the SC detector in term of the ROC is obtained and comparisons are made with Monte Carlo simulations as a check on the validity of the results.

## 2 Single-Cycle detector

The detection problem is stated in terms of the following binary hypothesis test:

$$\begin{aligned} H_0 &: x(t) = n(t), \\ H_1 &: x(t) = s(t) + n(t). \end{aligned} \quad t = 0, \dots, T-1 \quad (1)$$

where  $n(t)$  is a zero-mean stationary WGN process and  $s(t)$  is a zero-mean “weak” cyclostationary signal, which is assumed to be noise-independent.

When the signal  $s(t)$  is modeled as cyclostationary, the time-varying covariance of the discrete-time zero-mean process,  $C_{2s}(t, \tau) = E[s(t)s(t + \tau)]$ , can be expressed in a Fourier series as

$$C_{2s}(t, \tau) = \sum_{\alpha \in \mathcal{A}_{2s}} C_{2s}^\alpha(\tau) e^{j\alpha t} \quad (2)$$

$$C_{2s}^\alpha(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} C_{2s}(t, \tau) e^{-j\alpha t} \quad (3)$$

$$\text{with } \mathcal{A}_{2s} = \{\alpha : 0 \leq \alpha < 2\pi \text{ and } C_{2s}^\alpha(\tau) \neq 0\}, \quad (4)$$

and the Fourier coefficient  $C_{2s}^\alpha(\tau)$  is referred to as the cyclic covariance of  $s(t)$  at the cycle frequency  $\alpha$ .

The LO test and (2) lead to the MC statistic obtained in [2], from which by selecting one cycle frequency and taking the modulus to avoid SOI phases problem, the discrete-time single-cycle detector statistic is given by

$$\lambda = |Z| = \left| \sum_{\tau=-T+1}^{T-1} C_{2s}^\alpha(\tau) C_{2s}^{\alpha*}(\tau) \right| \quad \text{with } \alpha \neq 0, \quad (5)$$

where  $C_{2x_T}^\alpha(\tau)$  is the cyclic correlogram of the received data defined by

$$C_{2x_T}^\alpha(\tau) \triangleq \frac{1}{T} \sum_{t=0}^{T-\tau-1} x(t)x(t+\tau)e^{-j\alpha t} \text{ for } \tau \geq 0$$

$$C_{2x_T}^\alpha(\tau) = C_{2x_T}^\alpha(-\tau)e^{j\alpha\tau} \text{ for } \tau \leq 0. \quad (6)$$

### 3 Performance analysis

Some mixing conditions (which intuitively imply that samples of the process  $x(t)$  which are well separated in time are independent) defined in [1] assure the mean-square sense consistency of  $C_{2x_T}^\alpha(\tau)$  (i.e.  $\lim_{T \rightarrow \infty} E[C_{2x_T}^\alpha(\tau)] = C_{2x}^\alpha(\tau)$ ) and the asymptotic complex normality (ACN) of  $\sqrt{T}(C_{2x_T}^\alpha(\tau) - C_{2x}^\alpha(\tau))$ , with covariance expressions given in (10). Proofs are given in the appendix of [1].

From these results, the asymptotic distribution for the real and imaginary part of the statistic  $Z = Z_r + jZ_i$  (see 5) under  $H_0$  ( $x(t) = n(t)$ ) and  $H_1$  ( $x(t) = s(t) + n(t)$ ) is as follows<sup>1</sup>

$$\lim_{T \rightarrow \infty} \sqrt{T}(Z_{(r|i)} - E^\infty[Z_{(r|i)}]) = \mathcal{N}(0, V^\infty[Z_{(r|i)}]), \quad (7)$$

where  $\mathcal{N}$  stands for normal density and

$$E^\infty[Z_{(r|i)}] = \frac{1}{(2|2j)} \sum_{\tau} \{C_{2s}^\alpha(\tau)^* \lim_{T \rightarrow \infty} E[C_{2x_T}^\alpha(\tau)] + (|-)C_{2s}^\alpha(\tau) \lim_{T \rightarrow \infty} E[C_{2x_T}^\alpha(\tau)^*]\} \quad (8)$$

$$V^\infty[Z_{(r|i)}] = \frac{1}{2} \sum_{\tau, \rho} \Re\{C_{2s}^\alpha(\tau)^* C_{2s}^\alpha(\rho) S_{4x}^0(-\alpha)_{\tau, \rho} + (|-)C_{2s}^\alpha(\tau)^* C_{2s}^\alpha(\rho)^* S_{4x}^{2\alpha}(\alpha)_{\tau, \rho}\}. \quad (9)$$

In (9), the asymptotic covariance expressions [1] are given by

$$\lim_{T \rightarrow \infty} T \text{cov}\{C_{2x_T}^\alpha(\tau), C_{2x_T}^\alpha(\rho)^*\} = S_{4x}^0(-\alpha)_{\tau, \rho}$$

$$\lim_{T \rightarrow \infty} T \text{cov}\{C_{2x_T}^\alpha(\tau), C_{2x_T}^\alpha(\rho)\} = S_{4x}^{2\alpha}(\alpha)_{\tau, \rho} \quad (10)$$

where  $S_{4x}^\gamma(\omega)_{\tau, \rho}$  is the cyclic spectrum of  $f_{2x}(t; \tau)$ , defined by

$$S_{4x}^\gamma(\omega)_{\tau, \rho} \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{\xi=-\infty}^{+\infty} \text{cov}\{f_{2x}(t; \tau), f_{2x}(t+\xi; \rho)\} e^{-j\omega\xi} e^{-j\gamma t}, \quad (11)$$

where  $f_{2x}(t; \tau) = x(t)x(t+\tau)$  is a real valued time product.

From (7), and for  $T$  large enough, we may approximately express the distribution of the real and imaginary part of the statistic  $Z = Z_r + jZ_i$  under  $H_p$  ( $p = 0, 1$ ) as

$$Z_{(r|i)} \sim \mathcal{N}(E_p^\infty[Z_{(r|i)}], \frac{V_p^\infty[Z_{(r|i)}]}{T}). \quad (12)$$

<sup>1</sup> $Z_{(r|i)}$  denotes  $Z_r$  (resp.  $Z_i$ ) the real part of  $Z$  (resp. imaginary part of  $Z$ ).

By using the consistency of the cyclic correlogram of the received data of (1), conditional expected values of  $Z_r$  and  $Z_i$  statistics under  $H_p$  can be expressed as

$$H_0 : E_0^\infty[Z_{(r|i)}] = 0$$

$$H_1 : \begin{cases} E_1^\infty[Z_r] = \sum_{\tau=-T+1}^{T-1} |C_{2s}^\alpha(\tau)|^2 \\ E_1^\infty[Z_i] = 0 \end{cases} \quad (13)$$

Now, in order to have an analytic expression of variances (9), we need to evaluate the cyclic spectrum (11). Using the superimposition of the two independent zero-mean time-series ( $x(t) = s(t) + n(t)$ ), the cyclic spectrum (11) can be expressed as

$$S_{4x}^\gamma(\omega)_{\tau, \rho} = S_{4n}^\gamma(\omega)_{\tau, \rho} + S_{4s}^\gamma(\omega)_{\tau, \rho} + S_{2sn}^\gamma(\omega)_{\tau, \rho} + S_{2ns}^\gamma(\omega)_{\tau, \rho} + S_{snns}^\gamma(\omega)_{\tau, \rho} + S_{nssn}^\gamma(\omega)_{\tau, \rho} \quad (14)$$

where the general form of the cyclic cross-spectrum  $S_{abcd}^\gamma(\omega)_{\tau, \rho}$  is defined by

$$S_{abcd}^\gamma(\omega)_{\tau, \rho} \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{\xi=-\infty}^{+\infty} \text{cov}\{f_{ab}(t; \tau), f_{cd}(t+\xi; \rho)\} e^{-j\omega\xi} e^{-j\gamma t}, \quad (15)$$

where  $f_{ab}(t; \tau) = a(t)b(t+\tau)$  and  $f_{cd}(t; \tau) = c(t)d(t+\tau)$  are real-valued time products. Cyclic-spectra in (14) correspond to the general form  $S_{abcd}^\gamma(\omega)_{\tau, \rho}$  by substituting subscripts  $a, b, c, d$  for the correct SOI and noise subscripts  $s$  and  $n$ .

The cyclic spectrum of the time noise product  $f_{2n}(t; \tau)$  can be easily expressed as

$$S_{4n}^\gamma(\omega)_{\tau, \rho} = \sigma_n^4 \eta_\gamma (\delta_{\tau-\rho} + \delta_{\tau+\rho} e^{-j\tau\omega}), \quad (16)$$

where

$$\eta_\alpha \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} e^{-j\alpha t} = \begin{cases} 1 & \text{for } \alpha = 0 \text{ mod } 2\pi \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

is the Kronecker delta train.

The cyclic cross-spectra of the different time signal and noise products can be expressed as

$$S_{2sn}^\gamma(\omega)_{\tau, \rho} = \sigma_n^2 C_{2s}^\gamma(\tau - \rho) e^{-j\omega(\tau - \rho)} \quad (18)$$

$$S_{2ns}^\gamma(\omega)_{\tau, \rho} = \sigma_n^2 C_{2s}^\gamma(\rho - \tau) e^{j\gamma\tau} \quad (19)$$

$$S_{snns}^\gamma(\omega)_{\tau, \rho} = \sigma_n^2 C_{2s}^\gamma(\tau + \rho) e^{-j\omega\tau} \quad (20)$$

$$S_{nssn}^\gamma(\omega)_{\tau, \rho} = \sigma_n^2 C_{2s}^\gamma(-\rho - \tau) e^{j\omega\rho} e^{j\gamma\tau}. \quad (21)$$

Finally, the cyclic spectrum of the time signal product  $f_{2s}(t; \tau)$  can be expressed as

$$S_{4s}^\gamma(\omega)_{\tau, \rho} = \sum_{\xi=-\infty}^{\infty} e^{-j\omega\xi} (C_{4s}^\gamma(\tau, \xi, \xi + \rho) + \sum_{\beta \in \mathcal{A}_{2s}} e^{j(\gamma-\beta)\tau} \{C_{2s}^\beta(\xi) C_{2s}^{\gamma-\beta}(\xi + \rho - \tau) + C_{2s}^\beta(\xi + \rho) C_{2s}^{\gamma-\beta}(\xi - \tau)\}) \quad (22)$$

where  $C_{4s}^\gamma(\tau_1, \tau_2, \tau_3)$  is the 4th-order cyclic cumulant of the cyclostationary SOI defined in similar fashion as (3).

Variances  $V_0^\infty(Z_{(r|i)})$  under  $H_0$  can now be evaluated to a closed form: substitution of (16) into (9), yields

$$V_0^\infty(Z_r) = V_0^\infty(Z_i) = \sigma_n^4 \sum_{\tau=-T+1}^{T-1} |C_{2s}^\alpha(\tau)|^2. \quad (23)$$

Variances  $V_1^\infty(Z_{(r|i)})$  under  $H_1$  are evaluated by substitution of (14) into (9).

Notice that, under  $H_1$ , cyclic spectra for  $\gamma = 2\alpha$  can be identically zero (see (18)-(22)) and the principal cyclic spectrum related to the noise only<sup>2</sup> (16) is identically zero. Hence, from (9) we can approximate conditional variances under  $H_1$  as

$$V_1^\infty[Z_r] \simeq V_1^\infty[Z_i] = \frac{1}{2} \sum_{\tau, \rho} C_{2s}^\alpha(\tau)^* C_{2s}^\alpha(\rho) S_{4x}^0(-\alpha)_{\tau, \rho} \quad (24)$$

This approximation leads us to well-known distributions, when the magnitude  $\lambda = |Z_r + jZ_i|$  is taken :

- under  $H_0$ , the SC statistic follows a Rayleigh distribution and the corresponding analytical threshold  $\mathcal{T}_{(sc)}$  for a fixed false alarm probability  $P_{fa}$  can be evaluated by

$$\mathcal{T}_{(sc)}^2 = -\frac{\sigma_n^4}{T} \sum_{\tau=-T+1}^{T-1} |C_{2s}^\alpha(\tau)|^2 \ln(P_{fa}), \quad (25)$$

- under  $H_1$ , the SC statistic follows a Rice distribution and the corresponding detection probability is

$$P_d^{(sc)} = Q\left(\frac{\sum_{\tau=-T+1}^{T-1} |C_{2s}^\alpha(\tau)|^2}{\sqrt{\frac{V_1^\infty[Z_r]}{T}}}, \frac{\mathcal{T}_{(sc)}}{\sqrt{\frac{V_1^\infty[Z_r]}{T}}}\right), \quad (26)$$

where  $Q(a, b)$  is the Marcum function. An efficient algorithm of Marcum's  $Q$  function can be found in [4].

#### 4 An example : the ROC for BPSK signal

In order to form the ROC for the BPSK signal from the theoretical assessment in Sec.3, we need to evaluate the cyclic covariance and the fourth-order cyclic cumulant of a discrete BPSK time-series<sup>3</sup>.

##### 4.1 Second and fourth cyclic cumulant of a BPSK discrete-time series

Using the sampling frequency as an integer multiple of the baud rate of a continuous BPSK signal, the discrete-time BPSK signal  $s(t)$  can be expressed as:

$$s(t) = y(t)z(t) \quad (t \in \mathbb{Z}) \quad (27)$$

$$y(t) = \sum_{m=-\infty}^{\infty} a_m p(t + mT_c + t_0) \quad (28)$$

$$z(t) = A_s \cos(\omega_0 t + \theta), \quad (29)$$

<sup>2</sup>principal because of the "weak" signal assumption.

<sup>3</sup>Notice that the fourth order cyclic cumulant of the cyclostationary SOI appears only in (22).

where  $T_c$  corresponds to the number of points per keying interval,  $\omega_0$  is the reduced carrier pulsation,  $t_0$  is the normalized keying clock phase,  $\{a_m\}$  is an independent and identically distributed symbol sequence equal to  $-1, 1$  and  $p(t)$  is the rectangular pulse such that  $p(t) = 1$  for  $t = 0, \dots, T_c - 1$  and 0 otherwise.

The PAM process  $y(t)$  checks some "mixing conditions", so sample averages of a product of the PAM process converge with probability 1 to the time-average expectation of the product. This allows us to use the non-probabilistic results [5] about cyclic cumulant formulas for PAM time-series.

The  $n^{th}$ -order ( $n=2$  or 4) cyclic cumulant of the discrete PAM signal defined in (28) can be obtained from the discrete-time counterpart of cyclic cumulant formulas for PAM time-series in [5], and we can achieve the following closed-form expression

$$C_{ny}^\beta(\tau) = \frac{-2}{T_c} e^{-j\frac{\alpha+b}{2\beta}} \frac{\sin(\frac{\beta(b-a+1)}{2})}{\sin(\frac{\beta}{2})} e^{j\beta t_0} \quad (30)$$

$$\text{for } \beta = \frac{2\pi k}{T_c} \bmod 2\pi, \text{ with } k \in \{1, \dots, T_c - 1\}$$

$$C_{ny}^\beta(\tau) = \frac{-2}{T_c} (b - a + 1) \text{ for } \beta = 0 \bmod 2\pi \\ = 0 \text{ otherwise,} \quad (31)$$

where  $a = -\min\{0, \tau\}$ ,  $b = T_c - 1 - \max\{0, \tau\}$  and if  $a > b$  then  $C_{4y}^\beta(\tau) = 0$ .

We can easily compute the 2nd and 4th-order cyclic autocorrelation of the non random modulation  $z(t)$  with

$$R_{nz}^\gamma(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} z(t) \prod_{i=1}^{n-1} z(t + \tau_i) e^{-j\gamma t} \quad (32)$$

we obtain

$$R_{2z}^\gamma(\tau) = \frac{A_s^2}{4} \{2 \cos(\omega_0 \tau) \eta_\gamma + e^{j2\theta} e^{j\omega_0 \tau} \eta_{\gamma-2\omega_0} + e^{-j2\theta} e^{-j\omega_0 \tau} \eta_{\gamma+2\omega_0}\} \\ R_{4z}^\gamma(\tau) = A(\tau) \eta_\gamma + B(\tau) \eta_{\gamma-2\omega_0} + B(\tau)^* \eta_{\gamma+2\omega_0} + C(\tau) \eta_{\gamma-4\omega_0} + C(\tau)^* \eta_{\gamma+4\omega_0}$$

where

$$A(\tau) = \frac{A_s^4}{8} \{ \cos(\omega_0(\tau_1 + \tau_2 - \tau_3)) + \cos(\omega_0(\tau_1 - \tau_2 - \tau_3)) + \cos(\omega_0(\tau_1 - \tau_2 - \tau_3)) \}, \\ B(\tau) = \frac{A_s^4}{16} e^{2\theta} \{ e^{-j\omega_0(\tau_1 + \tau_2 + \tau_3)} + e^{-j\omega_0(\tau_1 + \tau_2 - \tau_3)} + e^{-j\omega_0(\tau_1 - \tau_2 + \tau_3)} + e^{-j\omega_0(-\tau_1 + \tau_2 + \tau_3)} \}, \\ C(\tau) = \frac{A_s^4}{16} e^{j4\theta} e^{-j\omega_0(\tau_1 + \tau_2 + \tau_3)}.$$

Finally, the cumulant of the product of an almost periodic deterministic modulated signal  $z(t)$  by a cyclostationary process  $y(t)$  is given by:

$$C_{ns}^\gamma(\tau) = \sum_{\beta \in \mathcal{A}_{ny}} C_{ny}^\beta(\tau) R_{nz}^{\gamma-\beta}(\tau), \quad (33)$$

where  $\mathcal{A}_{ny} = \{\frac{k2\pi}{T_c}; k = \{0, \dots, T_c - 1\}\}^4$

<sup>4</sup>Note that, the discrete PAM signal (28) has the same set of cycle frequency  $\mathcal{A}_{ny}$  for  $n = 2$  or  $n = 4$ .

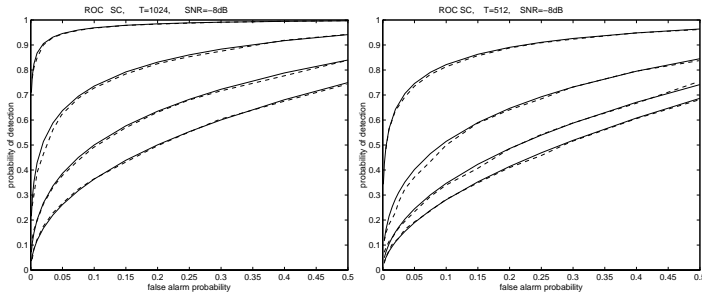


Figure 1: *ROC SC*,  $T = 1024$  and  $T = 512$

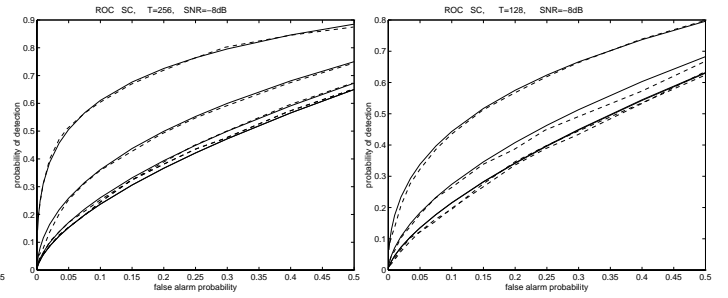


Figure 2: *ROC SC*,  $T = 256$  and  $T = 128$

## 4.2 Influence of the BPSK parameters on a correct implementation of the SC detector

Before to implement the SC statistic via computer simulations, we emphasize that the SOI parameters have a crucial impact on achieving effective implementation of the SC detector. In fact, to avoid unknown SOI phases (i.e. carrier phase  $\theta$  and the keying clock phase  $t_0$ ), the solution which consists in choosing one cycle frequency, and then taking the magnitude of the statistic doesn't always hold for the discretized BPSK signal model (27). According to the computation of the discrete cyclic covariance of a BPSK signal (see previous subsection), the summation (33) can be performed over several cycle frequencies  $\beta$ . This way, we see a sum of terms such as  $e^{\pm j2\theta}$  or  $e^{j\beta t_0}$  or  $e^{j(\beta t_0 \pm 2\theta)}$ . If there's only one term, this term could be eliminated when the modulus operation is performed to form the SC statistic. However, if there are several terms, the modulus operation cannot remove these terms, and this can give rise to destructive interference. To avoid this problem, the SC detector must be implemented such that  $\omega_0 T_c$  and  $2\omega_0 T_c$  aren't integer multiples of  $\pi$ .

## 4.3 Numerical results

Comparisons of analytical results with computer simulations (see (5)) for the ROC are presented for one value of the signal-to-noise-ratio (SNR) defined in the signal bandwidth as

$$SNR \triangleq 10 \log \frac{T_c \sigma_s^2}{4\sigma_n^2} = -8dB. \quad (34)$$

Parameters of the BPSK signal are: reduced pulsation  $\omega_0 = 2\pi f_0 = 2\pi \times 0.21$ , keying interval  $T_c = 4$  (four points per keying interval).

The ROC for the SC detector are evaluated for the selected cycle frequencies  $\frac{\alpha}{2\pi} = 2f_0$ ,  $1/T_c$ ,  $2f_0 - 1/T_c$  and  $-2f_0 + 2/T_c$ .

Figs. 1 and 2 plot the ROC for the each considerate cycle frequency and different sample size  $T$  (solid lines correspond to the theoretical results and dashed lines to the simulated results). Simulation results are based on a sample population of 3000. Performance of the SC detector are symbolically ordered for each cycle frequency as :  $SC(\frac{\alpha}{2\pi} = 2f_0) > SC(\frac{\alpha}{2\pi} = 1/T_c) > SC(\frac{\alpha}{2\pi} = 2f_0 -$

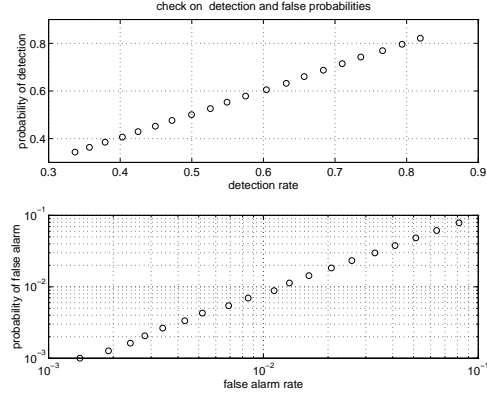


Figure 3: *Detection and false alarm probabilities*

$1/T_c) > SC(\frac{\alpha}{2\pi} = 2f_0 - 2/T_c)$ . We note that for  $T = 128$  we have  $SC(\frac{\alpha}{2\pi} = 2f_0 - 1/T_c) \approx SC(\frac{\alpha}{2\pi} = 2f_0 - 2/T_c)$ . These results confirm the good agreement between theoretical and simulated results for each sample time.

Fig. 3 plots i) theoretical probability of detection versus the estimated probability of detection and ii) theoretical probability of false alarms versus false alarms rate. We observe a good agreement between theoretical and estimated results. However, we note that we don't exactly have straight line from  $1e-3$  to  $1e-1$ , as expected. This can rise because the number of false alarms is very small for a 10000 sample population (for  $Pfa < 1e-2$ ).

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