On the approximation of nonbandlimited signals by nonuniform sampling series

Paulo Jorge S. G. Ferreira
Departamento de Electrónica e Telecomunicações / INESC
Universidade de Aveiro
3800 Aveiro Portugal
E-mail: pjf@inesca.pt

Abstract— **The classical WKS sampling theorem is a central result in signal processing, but it applies to band-limited signals only. For many purposes, this class of signals is too narrow. For example, the signals that occur in practice are invariably of finite duration, or time-limited, and often have discontinuities. Clearly, such signals cannot be band-limited. We consider the problem of approximating such signals, or other signals not necessarily band-limited, using sampling series. We do not assume that the sampling instants are regularly distributed, in order to account for errors due to jitter. To the best of our knowledge, the problem of obtaining nonuniform sampling approximations for signals not necessarily band-limited, despite its practical interest, has not been addressed in the literature. In this work we introduce a method that leads to sampling approximations with the required properties. It is shown that the sampling sums considered are capable of approximating a wide class of signals, with arbitrarily small L_2 and L_∞ errors.**

I. Introduction

A sampling series is a series of the form

$$f(t) = \sum_{k=-\infty}^{+\infty} f(t_k)\phi_k(t).$$

The points t_k are called the sampling points, and the function ϕ is called the kernel or the interpolating function. The WKS theorem asserts that any function f band-limited to w satisfies[1]

$$f(t) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k\pi}{w}\right) \frac{\sin[w(t-k\frac{\pi}{w})]}{w(t-k\frac{\pi}{w})}, \qquad (1)$$

the convergence being absolute and uniform. An elementary introduction to sampling theory may be found in [1]. The survey papers [2–5] and the books [6,7] give an account of the field, its history, and contain detailed bibliographies.

There are many important results concerning the approximation of not necessarily band-limited signals by sampling series. These results establish that certain classes of functions can be arbitrarily well approximated by sampling expansions, provided that the sampling period π/w is sufficiently small. An excellent and interesting overview of these results can be found in [5]. The review papers mentioned above, as well as [8–17], are among the works that somehow address this problem. We stress that these works consider uniform sampling only.

In this work we study nonuniform sampling expansions for not necessarily band-limited functions. The method used is an extension of the method introduced in [18], as presented in [19]. We study the approximation properties of such sampling series, having in mind the effect of jitter, and the behavior of the L_2 and L_∞ approximation errors, for specific kernels.

II. The sampling approximations

Let $f \in L_2$, and consider the convolution

$$f_w(t) = \int_{-\infty}^{+\infty} f(\tau)K(t-\tau)\,d\tau,$$

where the kernel K is a function of at least one parameter w, such that $f_w \to f$ as $w \to \infty$. There are many kernels K that share this property, the most common example being the sinc kernel

$$K(t) = \frac{\sin wt}{\pi t}. \qquad (2)$$

We will also consider

$$\begin{aligned} K(t) &= \frac{\cos(wt) - \cos[(w+a)t]}{\pi a t^2} \\ &= \frac{2\sin[(w+\frac{a}{2})t]\sin(at/2)}{\pi a t^2}, \end{aligned} \qquad (3)$$

and the sinc-squared kernel

$$K(t) = \frac{a}{2\pi}\left[\frac{\sin(at/2)}{at/2}\right]^2 = \frac{1-\cos(at)}{\pi a t^2}. \qquad (4)$$

In these cases f_w, which approaches f in the L_2 norm, also converges in the point-wise sense to f (provided that, for example, f has bounded variation).

The main idea of our method follows from the possibility of approximating $f \in L_2$ by f_w, to any prescribed tolerance, by taking w sufficiently large. The function f_w is then approximated by a nonuniform sampling series, whose coefficients are the samples of f itself.

Theorem 1: *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function of bounded variation vanishing outside* $[0,1]$*. Denote by* $\{t_i\}$ *any N reals such that*

$$\frac{i-1}{N} < t_i < \frac{i}{N}, \qquad (5)$$

[1] A function $f \in L_2(\mathbb{R})$ is band-limited to w if its Fourier transform vanishes outside $[-w, w]$.

for $1 \le i \le N$, *and let*

$$f_w(t) = \int_0^1 f(\tau) K(t-\tau)\, d\tau. \qquad (6)$$

Consider the sampling series

$$s_N(t) = \frac{1}{N} \sum_{k=1}^{N} f(t_k) K(t - t_k).$$

Then,

$$|f_w(t) - s_N(t)| \le \frac{V_f \|K\|_\infty + V_k(t) \|f\|_\infty}{N}, \qquad (7)$$

where $\|\cdot\|_\infty$ *denotes the* L_∞ *norm, and* $V_k(t)$ *denotes the variation of the function* $k(x) = K(x-t)$ *for* $x \in [0,1]$.

It turns out that, for the kernels mentioned above, $V_k(t)$ is bounded by a quantity that depends on w but not on t. This means that, given $\epsilon > 0$, it is possible to pick N such that $|f_w(t) - s_N(t)| < \epsilon$. Under these circumstances the sampling series yields a uniform approximation of f_w, to within ϵ.

It is a simple matter to deal with functions f whose support is contained in a compact interval $[a, b]$, other than $[0,1]$. It is slightly less straightforward, but nevertheless possible, to account for discontinuous signals f of bounded variation.

For the sinc kernel it can be shown that

$$\left| f_w(t) - \frac{1}{N} \sum_{k=1}^{N} f(t_k) \frac{\sin[w(t-t_k)]}{\pi(t-t_k)} \right| = \frac{O(w \log w)}{N}, \qquad (8)$$

for $0 \le t \le 1$. If t is outside $[0,1]$ the approximation is $O(w)/N$. If the kernel is of bounded variation, as the kernel (4), for example, the approximation becomes asymptotically better. In fact, for any t,

$$\left| f_w(t) - \frac{w}{2\pi N} \sum_{k=1}^{N} f(t_k) \left[\frac{\sin[w(t-t_k)]}{\pi(t-t_k)} \right]^2 \right| = \frac{O(w)}{N}. \qquad (9)$$

Similar results hold for other bounded variation kernels.

III. Approximation errors, jitter

In this section we investigate the error that arises when a signal f is approximated by the sampling sums considered above. The error is analyzed using both the L_2 and L_∞ norms.

Theorem 2: *Given* $\epsilon > 0$, *there exist* w *and* N *such that the sampling series* s_N *differs from* f *by less than* ϵ *in the* L_∞ *norm. This statement is true for any of the kernels mentioned.*

Proof: Clearly,

$$\begin{aligned} |f(t) - s_N(t)| &\le |f(t) - f_w(t)| + |f_w(t) - s_N(t)| \\ &\le |f(t) - f_w(t)| + \frac{O(w)}{N}. \end{aligned}$$

Take any two positive numbers α and β whose sum does not exceed ϵ. Now take w so large that

$$|f(t) - f_w(t)| \le \alpha,$$

and N so large that

$$|f(t) - s_N(t)| \le \beta.$$

Then

$$|f(t) - s_N(t)| \le \alpha + \beta \le \epsilon.$$

■

We now consider the approximation error in the L_2 norm in $[0,1]$. Recall that the norm is defined by $\|f\|^2 = \int_0^1 |f(t)|^2\, dt$. For simplicity we consider only the sinc kernel.

Theorem 3: *Given* $\epsilon > 0$, *there exist* w *and* N *such that the sampling approximation* s_N *(sinc kernel) differs from* f *by less than* ϵ *in the* L_2 *norm in* $[0,1]$.

Proof: In this case,

$$\|f - s_N\| \le \|f - f_w\| + \|f_w - s_N\|.$$

We evaluate each term separately. On one hand,

$$\begin{aligned} \|f - f_w\| &= \left(\int_0^1 |f(t) - f_w(t)|^2\, dt \right)^{1/2} \\ &< \left(\int_{-\infty}^{+\infty} |f(t) - f_w(t)|^2\, dt \right)^{1/2} \\ &\le \left(\int_{-\infty}^{+\infty} |\hat{f}(\omega) - \hat{f}_w(\omega)|^2\, d\omega \right)^{1/2} \\ &\le \left(2 \int_w^{\infty} |\hat{f}(\omega)|^2\, d\omega \right)^{1/2}, \end{aligned}$$

a quantity that tends to zero as $w \to \infty$. On the other hand, since

$$|f_w(t) - s_N(t)|^2 \le c^2 \frac{w^2 \log^2 w}{N^2},$$

it is true that

$$\begin{aligned} \|f_w - s_N\| &= \left(\int_0^1 |f_w(t) - s_N(t)|^2\, dt \right)^{1/2} \\ &\le c \frac{w \log w}{N}. \end{aligned}$$

Therefore

$$\|f - s_N\| \le \left(2 \int_w^{\infty} |\hat{f}(\omega)|^2\, d\omega \right)^{1/2} + c \frac{w \log w}{N}.$$

Take any two positive numbers α and β whose sum does not exceed ϵ. By taking w sufficiently large, the first term can be made smaller than any prescribed positive number α. Once w is fixed, proper choice of N can render the second term smaller than any $\beta > 0$. ■

We have some remarks to add to these results.

The first remark concerns the effect of jitter. It is felt only in the term

$$|f_w(t) - s_N(t)|,$$

for which the upper bounds given in the previous section are valid. For the sinc kernel, this term is $O(w \log w)/N$

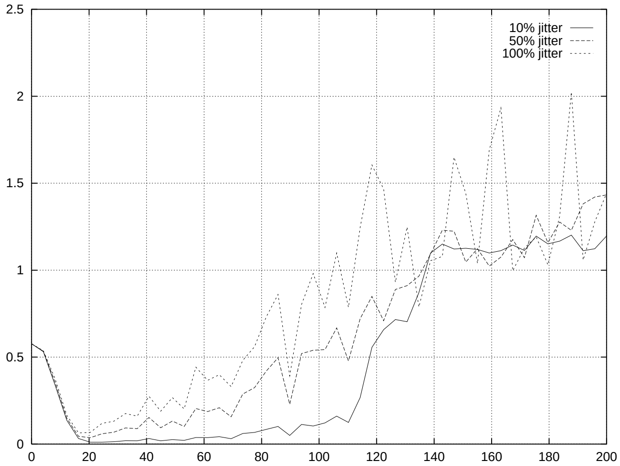

Fig. 1. L_∞ error versus bandwidth w. The sampling sum has a constant number of points and uses the sinc kernel.

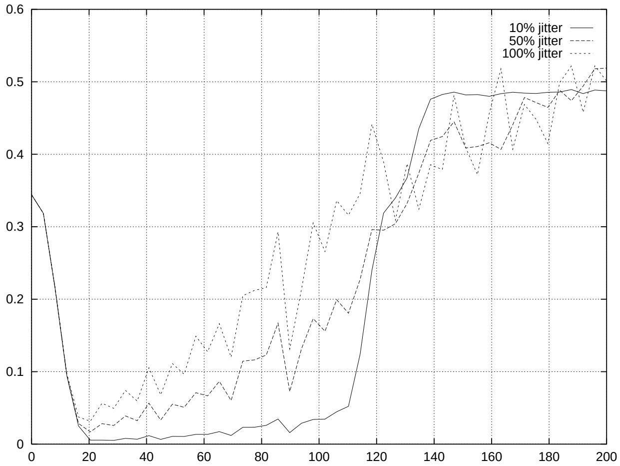

Fig. 2. L_2 error versus bandwidth w. The sampling sum has a constant number of points and uses the sinc kernel.

inside $[0,1]$, whereas for kernels of bounded variation it is $O(w)/N$. The effect of the jitter is not experimentally apparent unless N is comparatively low (as $N \to \infty$ the sum s_N approaches f_w uniformly).

The second remark is concerned with the behavior of the approximation error as function of w, for a fixed N. It follows from the previous results that the approximation error, measured using the L_∞ or the L_2 norms, achieves a minimum value as a function of w. Indeed, the total error is the sum of two components, one of which decreases when w increases, whereas the other is an increasing function of w. This behavior is clearly seen in figures 2-1.

The last remark has to do with the extension of the results given so far to signals g of bounded variation that do not necessarily vanish outside $[0,1]$. Can such signals be approximated by sampling sums of the type considered? The answer is yes. It is sufficient to take

$$h(t) = \begin{cases} g(t) & |t| \le R, \\ 0 & |t| > R, \end{cases}$$

that is, a truncated version of g. If R is sufficiently large, then $\|g-h\|$ is small in the L_2 norm. Now consider the signal defined by $f(t) = h[R(2t-1)]$ for $t \in [0,1]$, and $f(t) = 0$ for other t. The signal f can be approximated by a sampling sum like (8) or (9). The parameters N and w can always be selected to keep the L_2 approximation error less than any given ϵ. Since $h(t) = f[(t+R)/2R]$, and

$$f(t) \approx \frac{1}{N} \sum_{k=1}^{N} f(t_k) \frac{\sin[w(t-t_k)]}{\pi(t-t_k)},$$

it follows that

$$h(t) \approx \frac{2R}{N} \sum_{k=1}^{N} h(\tau_k) \frac{\sin[W(t-\tau_k)]}{\pi(t-\tau_k)},$$

where $\tau_k = 2Rt_k - R$, and $W = \frac{w}{2R}$. This concludes the argument, andshows how the sampling sums can be used to approximate L_2 functions of bounded variation.

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