MODULATED FILTER BANKS : A FOLDING APPROACH

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Abstract

Cosine Modulated Filter Banks (MFB) have been widely studied [MAL92, KOI92, MAU94] and are successfully used in signal and image processing. A perfect reconstruction factorization of filter banks based on cosine modulation of a linear phase prototype filter of length L=2KM has been proposed in [MAL92, KOI92]. This factorization leads to paraunitary MFB where the analysis and synthesis filters are the same. In this paper, we generalize the folding operator introduced in [AUC92]. That permits the extension of the factorization to the $L=NM$ case, with a more general modulation matrix. If N is even, the MFB is either paraunitary or biorthogonal. While if N is odd, the MFB is biorthogonal.

I The folding principle

Starting from a discrete orthonormal basis of M functions φ_k ($0 \le k \le M$-1) of M taps each, let us call Φ the MxM corresponding unitary transform whose each line k holds the M values $\varphi_k(n)$ ($0 \le n \le M$-1). One can extend this finite basis up to a function ψ_k having the suitable length ($L=NM$), using symmetry or antisymmetry at each bound (i.e. around n=0 and around $n=M$). The parity of the functions φ_k around n=0 is called the left parity and is denoted α (α=1 for a symmetry or α=-1 for an antisymmetry). Similarly, the parity of the functions around $n=M$ is called right parity and is denoted β.

This values are chosen to ensure a good regularity to the extended functions ψ_k. Fig. 1 illustrates this extension with $\alpha = -\beta = 1$ on a single function φ.

The extended functions are periodical with a period of at most 4M points and they have, thanks to their regularity, a narrow band spectrum. As an example, consider the DCT type IV unitary transform [RAO90]; the M basis functions may be extended and yield to a set of M pure harmonic functions if we choose (α, β)=(1,-1). Similarly for the DCT type II transform, we have (α, β) = (1,1).

Let us partition Φ as $\Phi=[\Phi_0\ \Phi_1]$ where Φ_0 and Φ_1 are MxM/2 matrices with an even M. The infinite duration extension of the matrix $\Phi=[\Phi_0\ \Phi_1]$ with parities (α,β) is given by (see Fig. 1) :

$$[\ldots\ \alpha\Phi_1 J\ \ \alpha\Phi_0 J\ \ \Phi_0\ \ \Phi_1\ \ \beta\Phi_1 J\ \ \beta\Phi_0 J\ \ \alpha\beta\Phi_0 \ldots] \quad (1)$$

where J is the square antidiagonal (or exchange) matrix of appropriate size. The product ΦJ is in inverse column order of the one of Φ.

Let us call the "bell" (or window) function $b(n)=h(L\text{-}1\text{-}n)$, $0 \le n \le L$-1, where $L=NM$, the inverse time order of the analysis prototype filter h [AUC92]. We split the bell into pieces containing each M/2 taps. We put each piece into a diagonal matrix: B_i=diag($b(iM/2)$, $b(iM/2+1)$, ..., $b(iM/2+M/2\text{-}1)$) where $0 \le i \le 2N$-1. Assuming the window function starts M/2 points before the basis, then from (1) we have for the M modulated functions the following MxNM matrix: [$\alpha\Phi_0 JB_0$ $\Phi_0 B_1$ $\Phi_1 B_2$ $\beta\Phi_1 JB_3$ $\beta\Phi_0 JB_4$ $\alpha\beta\Phi_0 B_5$ $\alpha\beta\Phi_1 B_6$ ...]. or under the product form:

$$\Phi.F' = \Phi\begin{bmatrix} \alpha J.B_0 & B_1 & 0 & 0 & \beta J.B_4 & \alpha\beta.B_5 & . & . \\ 0 & 0 & B_2 & \beta J.B_3 & 0 & 0 & . & . \end{bmatrix} \quad (2)$$

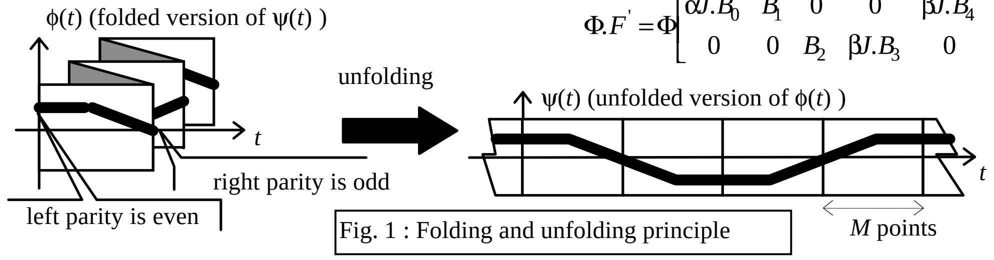

Fig. 1 : Folding and unfolding principle

The initial $M/2$ points offset corresponds to add a phase shift to the modulating functions. The use of M/2 points shift allows the particular zigzag structure (Fig. 2) and, as it will be shown later, ensure FIR filters as reconstruction bank.

The matrix F' is an MxNM sparse matrix F' and is called the folding template.

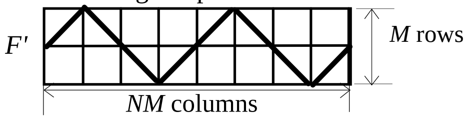

Fig. 2: Geometric plot of the folding matrix F' (N=4). (bold line follows the non zero values)

Let us call H the MxNM matrix holding the M time reversed impulse responses of the M FIR modulated filters. We use this finite matrix H as a copy template to fill an infinite banded matrix $\{H\}$ (Fig. 3). Notice the braces around the symbol H. They will be used in the following to express a similar infinite copy of a finite matrix.

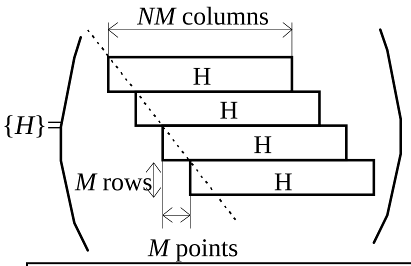

Fig. 3 : Analysis filter bank described by a block Toeplitz matrix.

To avoid border difficulties, let us call $\boldsymbol{x}$, the infinite time indexed column vector holding the input signal values $x(n)$, and let us call $\boldsymbol{y}$ an infinite column vector holding the decimated time indexed $\boldsymbol{y}_{iM}$ subvectors of M taps each. Here the kth entry of $\boldsymbol{y}_{iM}$ represents the scalar output of the kth filter at time iM. The M channel MFB is then described by : $\boldsymbol{y} = \{H\}\boldsymbol{x}$ [VET89]. The product given by (2) is equal to H, i.e.: $H=\Phi F'$. Thus, we have :

$$\{H\} = \{\Phi F'\} = \{\Phi\}\{F'\}$$

This latter writing corresponds to splitting the original filter bank into two cascaded processes: a so called folding stage $\{F'\}$ that maps NM points of the input signal to M points of the folded signal, followed by an MxM unitary transform Φ. $\{F'\}x$ is the N-1 time regularly folded signal. Notice that the matrix $\{\Phi\}$ is also unitary and block diagonal and then easily invertible ($\{\Phi\}^{-1} = \{\Phi^{T}\}$).

II Factorization:

<u>II-1 $L=2KM$ case (N even)</u>: To recover the input signal $\boldsymbol{x}$ from transformed signal $\boldsymbol{y}$, since $\{\phi\}$ is nonsingular, the inverse $\{F'\}^{-1}$ must exist. Because $\{F'\}$has an infinite size, we can read it as another matrix $\{F\}$, where F appears as a succession of "cross" matrices (Fig. 4). The Mx$(2K$-$1)M$ matrix F (the notation $F^{(K)}$ is avoided for simplicity) can be written as :

$$F=[\ F_0 \ \ 0 \ \ F_1 \ \ 0 \ \ ... \ \ F_{K\text{-}1}] \qquad (3)$$

where 0 are the MxM null matrix and F_j are given by:

$$F_j=(\alpha\beta)^j[B_{2+4j} \ \ \beta JB_{3+4j}\ ;\ \alpha JB_{4j} \ \ B_{1+4j}] \ \ 0\leq j\leq K\text{-}1 \qquad (4)$$

We use the semicolon ";" to indicate the ends of the rows of the matrix.

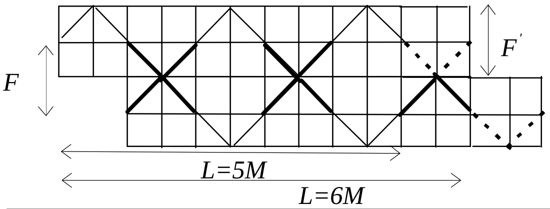

Fig. 4: Representation of the geometric plot of the matrices F' (zigzag matrix) and F (cross matrix) for N=5 and N=6, from two bands of $\{F'\}$.
Each little square is of size $M/2$x$M/2$.

Let us first review the L=2M case. As $F=F_0$, the matrix $\{F\}=\{F_0\}$ is block diagonal and easily reversed. For an arbitrary non singular F_0 matrix, The MFB is biorthogonal. To ensure linear phase of the analysis bell, that is, $B_2=JB_1J$ and $B_3=JB_0J$. Eq. (3) becomes:

$$F=[B_2 \ \ \beta B_0J\ ;\ \alpha JB_0 \ \ \ JB_2J\ ] \qquad (5)$$

If the product $\alpha\beta$=-1 we can choose:

$$B_2=\text{diag}(\cos(\theta_0), \cos(\theta_1),..., \cos(\theta_{M/2\text{-}1})) \qquad (6a)$$

$$B_0=\text{diag}(\sin(\theta_0), \sin(\theta_1),... \sin(\theta_{M/2\text{-}1})) \qquad (6b)$$

where the values of $M/2$ angles θ_i are arbitrary.

The matrix F is then unitary and define a paraunitary MFB. The inverse of the matrix $\{H\}$ is therefore given by $\{F\}^{-1}=\{F^{T}\}$ where the superscript T denotes transposition.

In the general case L=2KM, in order to be "close" to the matrix or (scalar) product, we use the MxM modified polyphase representation matrix $\underline{F}(z)$ for the folding stage $\{F\}$ defined by: $\underline{F}(z)=F(z^{-1})Jz^{-2(K-1)}$ where $F(z)$ is the MxM type 1 polyphase representation matrix of the folding stage $\{F\}$ [VET89, VAI93]. From (3) and (4) the matrix $\underline{F}(z)$ is simply:

$$\underline{F}(z) = F_0 + F_1 z^{-2} + ... + F_{K-1} z^{-2(K-1)}$$

Notice that the type 1 polyphase representation of the complete analysis filter bank is given by:

$$H(z)=z^{-2(K-1)}\Phi\,[0 \quad I\,;\, zI \quad 0]\underline{F}(z^{-1})J$$

where 0 and I are respectively the null and identity $M/2$ square matrices.

The fact that the product of two cross matrices is a cross matrix, allows the factorization of $\underline{F}(z)$. Let $\underline{F}^{(K)}(z)$ be such matrix where we have added the superscript K to indicate the length of the corresponding prototype filter. Since this matrix contains only the even power of z^{-1}, we can get $\underline{F}^{(K+1)}(z)$ by (post-multiply for example) :

$$\underline{F}^{(K+1)}(z) = \underline{F}^{(K)}(z)\Lambda(z)T^{(K)} \qquad (7a)$$

$$\Lambda(z) = [\, I \;\; 0 \;\; ; 0 \;\; z^{-2}I \,] \qquad (7b)$$

where $T^{(K)}$ is an arbitrary MxM cross matrix defined by $2M$ arbitrary parameters, 0 and I are the $M/2$ null and identity matrices. The matrix $\underline{F}^{(K+1)}(z)$ has the same structure as the one given in (3) and it is composed of K cross matrices.

From Eqs. (7), we can write:

$$\underline{F}^{(K)}(z) = \underline{F}^{(1)}.\prod_{j=1}^{K-1}\Lambda(z).T_j^{(K)} \qquad (8)$$

where $\underline{F}^{(1)}$ is an arbitrary constant cross matrix.

If all the constant matrices in (8) are nonsingular, the MFB is biorthogonal and the synthesis filter bank is simply obtained from the inverse of $\underline{F}^{(K)}(z)$:

$$(\underline{F}^{(K)})^{-1}(z) = (\prod_{j=1}^{K-1}(T_j^{(K)})^{-1}\Lambda(z^{-1})).(\underline{F}^{(1)})^{-1} \qquad (9)$$

The unfolded operator is defined from its modified polyphase representation given by :

$$\underline{U}^{(K)}(z)=z^{-m}\,(\underline{F}^{(K)}(z))^{-1}$$

with $m=2(K\text{-}1)$ so that $\underline{U}^{(K)}(z)$ contains only the negative even power of z^{-1}.

One can easily verify that $\underline{U}^{(K)}(z)$ inherits the cross structure from $\underline{F}^{(K)}(z)$. The synthesis filters are then modulated FIR filters.

The filter bank becomes paraunitary if we force all constant matrices in (9) to be unitary. Therefore the analysis and synthesis prototypes are flipped version of each other [VAI93].

A useful property is the linear phase of the prototype filter, which is given by the following relation: $B_j = JB_{4K-1-j}\; J$ $(0 \leq j \leq 4K\text{-}1)$. It can be shown that if $T_j^{(K)}$ matrices are as follows:

$$T_j^{(K)}=[E_j^{(K)} \quad C_j^{(K)}J \;\; ; \;\; JC_j^{(K)} \quad \alpha\beta JE_j^{(K)}J] \qquad (13)$$

then the prototype filter is linear phase [KOI92, MAL92].

With (13) each matrix $T_j^{(K)}$ is defined by only M extra parameters contained in the $M/2$x$M/2$ diagonal matrices $E_j^{(K)}$ and $C_j^{(K)}$.

If $\alpha\beta$=-1 we can impose the $T_j^{(K)}$ to be unitary by setting the forms (6a) for $E_j^{(K)}$ and (6b) for $C_j^{(K)}$. The matrix $T_j^{(K)}$ is then defined by only $M/2$ independent parameters. The $F^{(1)}$ matrix must keep the cross structure (5, 6a and 6b). The resultant analysis filter bank will be paraunitary.

A factorization similar to (8) with the constraints (5) and with unitary matrices $T_j^{(K)}$ was given in references [MAL92, KOI92], where the modulation matrix is the DCT IV matrix ($\alpha\beta$=-1).

In the equal parity case ($\alpha\beta$=1), the $T_j^{(K)}$ matrices (10) can't be unitary, the paraunitary property of the MFB is lost: the filter bank becomes biorthogonal. Like in $L=2M$ case [JAW94], an annoying property appears : the synthesis prototype filter loses its smoothness.

<u>II-2 *L*=(2*K*+1)*M* case (*N* odd)</u>: The matrix F is Mx$(2K+1)M$ and is given by : $[F_0 \quad 0 \quad F_1 \quad 0 \; ... \; F_{K-1} \quad 0 \quad F_K]$ where each F_j has a cross form, the same form as in the $L=2KM$ case except the last F_K which is given by (see Fig. 4):

$$F_K=(\alpha\beta)^K[0 \;\; 0\; ;\; \alpha JB_{4K} \;\; B_{1+4K}\,]$$

The results for the even case remain valid with the following modifications:

- we replace K by K+1 everywhere
- In Eq. (8 and 9), the initial matrix $F^{(1)}$ is now replaced by the triangular matrix: $F^{(0)}=[E^{(0)} \;\; 0\; ;\; C^{(0)} \;\; D^{(0)}]$. This form is necessary to generate the F_K matrix.
- it can be shown that the prototype cannot be linear phase. Hence Eq (10) is no longer necessary.

The triangular form of the $F^{(0)}$ matrix implies that the filter bank can't be paraunitary but is biorthogonal.

III Examples

In the paraunitary case, the design of a MFB is derived from the design of the analysis bell. A traditionally used objective function is a quadratic measure of stopband attenuation of the lowpass prototype filter [KOI92, MAL92].

In the biorthogonal case, where the analysis bell is different from the synthesis one, we have noticed that the analysis and synthesis prototypes are well designed if we use the same objective function as in the paraunitary case.

Fig. 5 shows a plot of a 2M length lowpass prototype filter of a 8-channel paraunitary MFB and the corresponding magnitude response.

Fig. 6 illustrates the case of a 3M length analysis lowpass prototype filter of a 8-channel biorthogonal MFB. The corresponding synthesis prototype filter is not represented here, because it has a similar shape as the analysis one. With a judicious choice of the $T_j^{(K)}$, we can impose equality between analysis and synthesis bells.

IV Conclusion:

In this paper, a new formulation of the MFB was presented where the modulation matrix is only constrained to have some symmetries. A factorization of the MFB which takes advantage of the geometric structure of the folding operator was given, where the prototype is of length $L=NM$. The MFB can be paraunitary only if N is even. By relaxing this property, we have much degree of freedom to design the prototype filter for given objective specifications, and a given prototype length.

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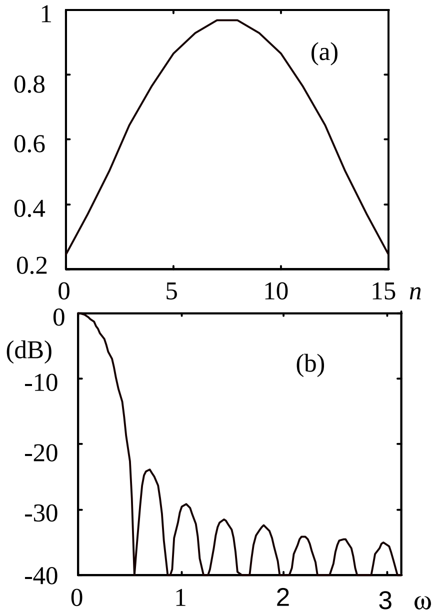

Fig. 5 Prototype filter for $L=2M$ and $M=8$.
(a) Impulse response.
(b) Magnitude response.

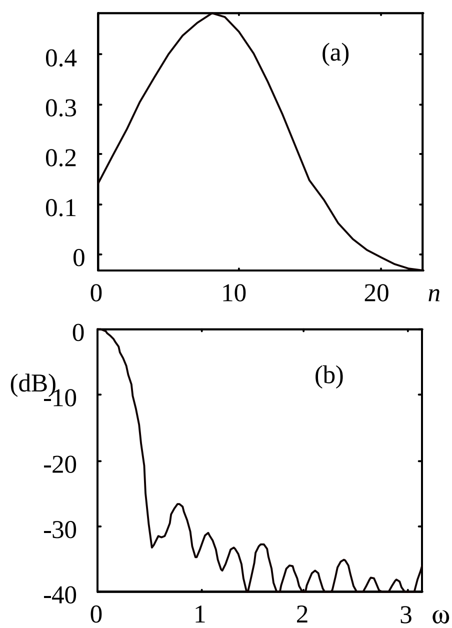

Fig. 6 Analysis prototype for $L=3M$ and $M=8$.
(a) Impulse response.
(b) Magnitude response.