AN ALGORITHM FOR ROBUST STABILITY OF DISCRETE SYSTEMS

M. Barret† and M. Benidir‡

† Supélec, Campus de Metz, 2 rue E. Belin 57070 Metz, France,

tel: (33) 87 74 99 38, fax: (33) 87 76 95 49, e-mail: Michel.Barret@supelec.fr

‡ Université Paris-Sud, L2S-ESE, Plateau de Moulon, 91192 Gif-sur-Yvette Cédex, France,

tel: (33) 1 69 85 17 17, fax: (33) 1 69 85 12 34, e-mail: benidir@lss.supelec.fr

ABSTRACT

In many applications digital recursive filter coefficients have no distinct values, therefore the test of an entire family of polynomials is required in order to be sure of the filter stability. The edge theorem by Bartlett, Hollot and Lin states that a polytope is stable, if and only if, the exposed edges are stable. In this paper this last condition is transformed into an equivalent one that can be tested in a finite number of arithmetic operations and from which an algorithm is derived. It is shown that the condition, which has been established, is optimal i.e., it can neither be avoided, nor simplified.

1 INTRODUCTION

Continuous or digital recursive filters are used in many applications in automatic control and signal processing. The study of the stability of such filters results in finding the location of the zeros of a polynomial, which is the transfer function denominator. In order to be sure of the asymptotic stability of such a filter, it is necessary and sufficient for the zeros of this polynomial to be all located in a distinct domain of the complex plane. This domain is the open unit-disk

$$U = \{z \in \mathbf{C} : |z| < 1\} \tag{1}$$

for a digital filter and the open half-plane

$$H = \{z \in \mathbf{C} : \Re(z) < 0\} \tag{2}$$

limited by the imaginary axis for a continuous filter. There are algorithms which decide (i.e., answer by either yes or no in a finite number of steps) whether all the zeros of a complex polynomial are either in H [1–2] or in U [3].

In many applications the filter coefficients have no distinct values and in order to be sure of the filter stability, the test of the zero location of an entire family of polynomials is required. When the filter is continuous and the family defined by the real polynomials

$$a_0x^n + \cdots + a_n \text{ with } a_i \in [\underline{a}_i\,;\,\overline{a}_i] \quad (0 \leq i \leq n), \tag{3}$$

where $\underline{a}_i \leq \overline{a}_i$ $(0 \leq i \leq n)$ are real numbers, a theorem by Kharitonov [4] states that all the zeros of all the polynomials (3) are each in H, if and only if, four of them satisfy the condition. It is then easy to establish an algorithm which decides in a finite number of steps whether all the elements of a family of continuous filters associated with the polynomials (3), are stable. The Kharitonov theorem—which can be extended to complex polynomials—has no equivalent statement when the half-plane H is replaced by the unit-disk U [9]. For the study of the robust stability of digital recursive filters, one can use the edge theorem of Bartlett, Hollot and Lin [5]. Applied to the domain U, this theorem states that all the zeros of all the elements of a polynomial polytope (i.e., the convex hull of many finitely generated polynomials $P, Q, \ldots, T$) are in U, if and only if, all the zeros of all the polytope edge elements are in U; i.e., if and only if, for any vertices P, Q of the polytope, all the zeros of all the polynomials

$$S_\lambda = \lambda P + (1-\lambda)Q \quad \text{for} \quad \lambda \in [0\,;\,1] \tag{4}$$

are in U. There are as many polynomials (4) as real numbers in the segment $[0\,;\,1]$ and, as far as we know, no paper exists which indicates if this last condition can be tested in a finite number of steps. However, this is necessary for an implementation on a computer. In this paper, it is shown that all the zeros of all the polynomials in an edge—defined in relation (4)—are in U, if and only if, one of both vertices P, Q satisfies this condition and

$$\mathbf{R}(\lambda) \neq 0 \qquad \text{for all} \qquad \lambda \in [0\,;\,1], \tag{5}$$

where $\mathbf{R}(\lambda)$ is a real polynomial in λ, which can be deduced in a finite number of steps from the coefficients of polynomials P and Q. This is the subject of section 3. The condition (5) can be decided using Sturm's theorem [8]. Using a previous work [6], it is shown that condition (5) is optimal when the coefficients of vertices P, Q can take any complex value *a priori*. To be precise, condition (5) can neither be avoided, nor simplified : any algorithm, which decides whether all the zeros of all the polynomials (4) are in U, must compute the polynomial $\mathbf{R}(\lambda)$—or a polynomial in λ which is a multiple of

$\mathbf{R}(\lambda)$ in the space of real or complex polynomials in λ—and test whether it vanishes on the real segment $[0\,;\,1]$. When both vertices P, Q are real, condition (5) is not optimal any more : the polynomial $\mathbf{R}(\lambda)$ can be factorized *a priori*, as we shall see in section 3. A complete algorithm for robust stability is given in section 4.

Let us specify notations and let us recall some results of previous works used below.

2 RECALLS AND NOTATIONS

Let us introduce a general polynomial with complex coefficients

$$P = a_0x^n + a_1x^{n-1} + \cdots + a_n \tag{6}$$

of a degree which is not greater than n. In what follows, the coefficient of highest degree a_0 may vanish. Let P^* be the polynomial deduced from P according to :

$$P^* = \overline{a}_nx^n + \overline{a}_{n-1}x^{n-1} + \cdots + \overline{a}_0, \tag{7}$$

where $\overline{a}_k$ denotes the complex conjugate of a_k. The polynomial P^* is called the conjugate reciprocal of P when P is of degree n, i.e., $a_0{\neq}0$. By convention, when the degree k of P is smaller than n, we shall say that infinity (∞) is a zero of P with a multiplicity $n-k$. With this convention, each polynomial P defined by relation (6) and not identically null has n zeros, including the multiplicities and the zeros at infinity. The polynomial P is associated with the single point of $\mathbf{C}^{n+1}$, also denoted by P, having $(a_0, \ldots, a_n)$ as coordinates. Let $\mathcal{D}_n$ be the subset of the space $\mathbf{C}^{n+1}$, which contains all the points P, whose all roots of the associated polynomial belong to the open unit-disk U. Let us introduce the resultant $\mathbf{R}(P, P^*)$ of polynomials P and P^* [6-7]. It can be expressed as a determinant of order $2n$ given by the relation

$$\mathbf{R}(P,P^*) = \begin{vmatrix} a_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \overline{a}_n \\ a_1 & a_0 & \ddots & \vdots & \vdots & .\cdot{}^{\cdot} & \overline{a}_n & \overline{a}_{n-1} \\ \vdots & \vdots & \ddots & 0 & 0 & .\cdot{}^{\cdot} & \vdots & \vdots \\ a_n & a_{n-1} & \ddots & a_0 & \overline{a}_n & .\cdot{}^{\cdot} & \overline{a}_1 & \overline{a}_0 \\ 0 & a_n & \ddots & a_1 & \overline{a}_{n-1} & .\cdot{}^{\cdot} & \overline{a}_0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & .\cdot{}^{\cdot} & .\cdot{}^{\cdot} & \vdots \\ 0 & \cdots & 0 & a_n & \overline{a}_0 & 0 & \cdots & 0 \end{vmatrix}. \tag{8}$$

It is well-known that when $a_0{\neq}0$ or $\overline{a}_n{\neq}0$, the polynomials P and P^*, defined by relation (6) and (7), have a common zero, if and only if, their resultant $\mathbf{R}(P, P^*)$ vanishes. In this case, the common zero is finite. Moreover, with the above-mentioned convention about zeros at infinity, if $a_0 = a_n = 0$, then polynomials P and P^* vanish each at infinity and $\mathbf{R}(P, P^*) = 0$.

Let us recall two propositions given in [6] and [7], which will be useful in the following section.

Proposition 1 ([7]) *For any complex polynomial defined by relation (6), the resultant $\mathbf{R}(P, P^*)$ is real.*

Let us assume now that the polynomial coefficients a_k $(0{\leq}k{\leq}n)$ introduced in equation (6) are each a continuous function of the real variables $v = (v_1, \ldots, v_r) \in I$, where $I = I_1 \times I_2 \times \cdots \times I_r$ is the cartesian product of non empty real intervals. Let $(P_v)_{v\in I}$ be the family of all the polynomials obtained with these continuous functions a_k $(0{\leq}k{\leq}n)$, when $v \in I$. For a fixed value of $v \in I$, let P_v^* be the polynomial deduced from P_v by the relation (7), when the coefficients a_k $(0{\leq}k{\leq}n)$ are replaced by the complex values $a_k(v)$ $(0{\leq}k{\leq}n)$ of the functions a_k $(0{\leq}k{\leq}n)$ at the point v.

Proposition 2 ([6]) *Let $(P_v)_{v\in I}$ be the above-mentioned family of polynomials. Every polynomial of $(P_v)_{v\in I}$ belongs to $\mathcal{D}_n$, if and only if, there exists $v_0 \in I$ such that $P_{v_0} \in \mathcal{D}_n$ and the resultant $\mathbf{R}(P_v, P_v^*)$ of P_v and P_v^*, which is a continuous function of v, does not vanish for all $v \in I$.*

3 A CRITERIA FOR ROBUST STABILITY

Let $\mathbf{P}$ be a polynomial polytope generated by the many finitely polynomials $P, Q, \ldots, T$. On the one hand, the application of the edge theorem to the open unit-disk U gives :

Theorem 3 (Bartlett, Hollot and Lin [5]) *A polytope is stable, if and only if, its exposed edges are stable.*

In other words, $\mathbf{P} \subset \mathcal{D}_n$, if and only if, for any vertices P, Q of the polytope, all the polynomials S_λ introduced in relation (4) belong to $\mathcal{D}_n$. On the other hand, the family of polynomials (S_λ) $(\lambda \in [0\,;\,1])$ satisfies the conditions of Proposition 2. Therefore, each polynomial S_λ $(\lambda \in [0\,;\,1])$ belongs to $\mathcal{D}_n$, if and only if, P (or Q) is in $\mathcal{D}_n$ and the resultant $\mathbf{R}(\lambda) = \mathbf{R}(S_\lambda, S_\lambda^*)$ of S_λ and S_λ^* does not vanish for all $\lambda \in [0\,;\,1]$, where S_λ^* is deduced from S_λ as the polynomial P^*—defined in relation (7)—is deduced from the polynomial P, introduced in equation (6). Moreover, if n denotes the highest degree of both vertices P, Q, it results from relation (8) and Proposition 1 that $\mathbf{R}(\lambda)$ is a real polynomial in λ, whose degree is not greater than $2n$. The next proposition is then set.

Proposition 4 *A polytope edge—defined in relation (4)—is stable, if and only if, one vertex is stable and the above-mentioned polynomial $\mathbf{R}(\lambda)$ does not vanish on the real segment $[0\,;\,1]$.*

In order to test condition (5), one can use Sturm's theorem [8]. Using the generalized Levinson-Szegø algorithm, the value of $\mathbf{R}(\lambda)$ can be fastly computed when a real value is assigned to λ [7]. Then, the coefficients of the polynomial $\mathbf{R}(\lambda)$ can be computed by interpolation [10] from $2n+1$ numerical values of $\mathbf{R}(\lambda)$.

The next proposition indicates that if the coefficients of vertices P, Q can take any complex value *a priori*, then the above-mentioned resultant $\mathbf{R}(\lambda)$ cannot be factorized *a priori*. We shall see at the end of section 3 that this fact is wrong when all the coefficients of both vertices are real.

Proposition 5 *Let P be defined by relation (6) and $Q = b_0x^n + \cdots + b_n$ be two complex polynomials whose degrees are not greater than n. Let α, β and γ, δ be the real and imaginary parts of a and b respectively :*

$$a_k = \alpha_k + i\beta_k \text{ and } b_k = \gamma_k + i\delta_k \quad (0\leq k\leq n). \tag{9}$$

Let λ be a real parameter. Let S_λ be the polynomial defined in relation (4) and let S_λ^* be the polynomial deduced from S_λ by the relation

$$\begin{aligned} S_\lambda^* &= \lambda P^* + (1-\lambda)Q^* \qquad (10)\\ &= [\lambda(\overline{a}_n - \overline{b}_n) + \overline{b}_n]x^n + \cdots + \lambda(\overline{a}_0 - \overline{b}_0) + \overline{b}_0. \end{aligned}$$

When all the coefficients of polynomials P, Q and λ are indeterminate, the resultant $\mathbf{R}(\lambda)$, obtained by eliminating x between the equations $S_\lambda = 0$ and $S_\lambda^* = 0$, is an irreducible polynomial in the variables $\alpha_0, \ldots, \alpha_n$, $\beta_0, \ldots, \beta_n$, $\gamma_0, \ldots, \gamma_n$, $\delta_0, \ldots, \delta_n$ and λ, with integer coefficients.

The proof of this proposition, whose main steps are given in Appendix, is based on the irreducibility of the polynomial $\mathbf{R}(P, P^*)$ in the space of real polynomials whose variables are the real and imaginary parts of the coefficients of P [6].

In [6] it is shown that condition (5) cannot be avoided : any algorithm, which decides whether each polynomial S_λ—defined in relation (4)—belongs to $\mathcal{D}_n$ is equivalent to an algorithm that tests if condition (5) is satisfied. Proposition 5 indicates that the inequality in relation (5) cannot be simplified *a priori*. When all the coefficients of both vertices P, Q real real, the above-mentioned resultant $\mathbf{R}(\lambda)$ can be factorized *a priori*. The equations

$$\begin{aligned} (1+x)^n P((1-x)/(1+x)) &= p(x^2) + xq(x^2) \qquad (11)\\ (1+x)^n Q((1-x)/(1+x)) &= r(x^2) + xs(x^2) \qquad (12) \end{aligned}$$

give four real polynomials p, q, r, s. The transformation which associates the coefficients of polynomials p and q with the ones of polynomial P is linear and depends only on the integer n. Let f_λ and g_λ be the polynomials associated, *via* this linear transformation, with S_λ—introduced in relation (4)—. They satisfy the following equalities :

$$\begin{aligned} f_\lambda &= \lambda p + (1-\lambda)r \qquad (13)\\ g_\lambda &= \lambda q + (1-\lambda)s. \qquad (14) \end{aligned}$$

It can be easily deduced from the relation—given in [6]—

$$(-1)^{n(n+1)/2}P(1)P(-1)(\mathbf{R}(p,q))^2 = 2^{n(n-1)}\mathbf{R}(P, P^*), \tag{15}$$

where $\mathbf{R}(p, q)$ is the resultant of polynomials p and q, that, if both vertices P, Q are real, then

$$\mathbf{R}(\lambda) = \mu_n S_\lambda(1) S_\lambda(-1)(\mathbf{R}(f_\lambda, g_\lambda))^2, \tag{16}$$

where $\mu_n = (-1)^n(-1/4)^{n(n-1)/2}$ and $\mathbf{R}(f_\lambda, g_\lambda)$ is the resultant obtained by eliminating x between both equations $f_\lambda = 0$ and $g_\lambda = 0$.

When all the coefficients of both vertices P, Q are real and indeterminate, it can be shown that the resultant $\mathbf{R}(f_\lambda, g_\lambda)$ is an irreducible polynomial in the variables $a_0, \ldots, a_n$, $b_0, \ldots, b_n$ and λ, with integer coefficients.

4 AN ALGORITHM FOR ROBUST STABILITY

Let $\mathbf{P}$ be a polynomial polytope generated by the many finitely complex polynomials $P_1, P_2, \ldots, P_m$. The following algorithm decides whether all the polynomials in $\mathbf{P}$ are stable. Let deg$P$ denote the degree of the polynomial P.

```
Inputs : P_1, P_2, ..., P_m.
For i = 1 to m do :
    {
    Test if P_i is stable,
        else print "the polytope is unstable", end.
    For j = i + 1 to m do :
        {
        n = max(degP_i, degP_j)
        For k = -n to n do :
            {
            Compute α_k = R(k), where R(λ) is the
            resultant, introduced in the previous sec-
            tion, of polynomials λP_i + (1 - λ)P_j and
            λP_i* + (1 - λ)P_j*. This can be done fastly
            with the Levinson-Szegø algorithm [7].
            }
        Compute by interpolation [10] the coefficients
        of polynomial R(λ), whose degree is not grea-
        ter than 2n, from the 2n + 1 real values α_k
        (-n≤k≤n).
        Test if R(λ)≠0 for all λ ∈ [0 , 1], using
            Sturm's theorem [8],
          else print "the polytope is unstable", end.
        }
    }
Print "the polytope is stable", end.
```

Exemple 1. Let us consider the polynomial polytope generated by both complex polynomials

$$\begin{aligned} P &= 2x^3 + (6/5 - 2i)x^2 + 1 \qquad (17)\\ Q &= 2x^3 + (6/5 + 2i)x^2 + 1. \qquad (18) \end{aligned}$$

The polynomial Q is deduced from the polynomial P by conjugating its coefficients. The roots of P, which are roughly equal to

$$x = -0.82261 + 0.5236i, \quad |x| = 0.97512 \tag{19}$$

$$x = -0.04564 + 0.9464i, \quad |x| = 0.94750 \quad (20)$$
$$x = +0.26827 - 0.4700i, \quad |x| = 0.54117 \quad (21)$$

are each in U, as those of Q which are the conjugates of the roots of P. The resultant $\mathbf{R}(\lambda)$ computed by the algorithm is given by

$$\mathbf{R}(\lambda) = -256\lambda^4 + 512\lambda^3 - \frac{6432}{25}\lambda^2 + \frac{32}{25}\lambda + \frac{539}{625}, \quad (22)$$

and vanishes twice in the segment $[0\,;\,1]$: its roots are roughly equal to the elements of the set

$$\Lambda = \{-0.0528\,;\,0.0648\,;\,0.9352\,;\,1.0528\} \quad (23)$$

Exemple 2. Let us consider the polynomial polytope generated by both real polynomials $P = 2x^3 - x^2 + 1$ and $Q = -2x^3 - x^2 + 1$, that satisfy the relation $P(-x) = Q(x)$. The roots of P are approximately given by

$$x = -0.6573, \quad (24)$$
$$x = 0.5787 \pm 0.6526i, \quad |x| = 0.8722. \quad (25)$$

The resultant $\mathbf{R}(\lambda)$ computed by the algorithm is equal to

$$\mathbf{R}(\lambda) = 16(-1+2\lambda)^2(1-8\lambda+8\lambda^2)^2 \quad (26)$$

and has six zeros, with their multiplicity, in the segment $[0\,;\,1]$. Each root of $\mathbf{R}(\lambda)$ is of multiplicity two, they are the elements of the set $\Lambda = \{0.5\,;\,0.1464\,;\,0.8536\}$.

5 APPENDIX

Proposition 5 is a specific case of the lemma below. In the following, when $f, g, \ldots, h$ denote indeterminates, the space of polynomials, whose variables are these indeterminates and whose coefficients are rational numbers, is denoted by $\mathbf{Q}[f, g, \ldots, h]$.

Lemma 6 *Let $\mathbf{R}(\alpha_0, \ldots, \alpha_n)$ be an irreducible polynomial in $\mathbf{Q}[\alpha_0, \ldots, \alpha_n]$. Let $a_0, \ldots, a_n$, $b_0, \ldots, b_n$ and λ be indeterminate linked to the α_i's by the relations*

$$a_i\lambda + b_i = \alpha_i \qquad (0 \le i \le n). \quad (27)$$

When $a_i\lambda + b_i$ $(0 \le i \le n)$ is substituted for α_i in $\mathbf{R}(\alpha_0, \ldots, \alpha_n)$ $(0 \le i \le n)$, this gives a polynomial $\mathbf{S}$ in the variables $a_0, \ldots, a_n$, $b_0, \ldots, b_n$ and λ :

$$\mathbf{S}(a_0, \ldots, a_n, b_0, \ldots, b_n, \lambda) = \mathbf{R}(a_0\lambda + b_0, \ldots, a_n\lambda + b_n) \quad (28)$$

which is irreducible in $\mathbf{Q}[a_0, \ldots, a_n, b_0, \ldots, b_n, \lambda]$.

Proof. Let us suppose that $\mathbf{S}$ is not irreducible, then there are two polynomials $\mathbf{S}_1$ and $\mathbf{S}_2$ in the space $\mathbf{Q}[a_0, \ldots, a_n, b_0, \ldots, b_n, \lambda]$, which are not constant and such that $\mathbf{S} = \mathbf{S}_1\mathbf{S}_2$. The proof is based on the following idea. If λ is set to an unknown value, i.e., is a parameter, if for $0 \le i \le n$, either a_i or b_i is a parameter and if all the $n+1$ other variables are indeterminates, then both polynomials S_1 and S_2 can be considered as polynomials in the variables $\alpha_0, \ldots, \alpha_n$, using the relations (27). The assumption that $\mathbf{S}$ is irreducible leads to the fact that either it is $\mathbf{S}_1$ or $\mathbf{S}_2$ that does not depend on the indeterminates but only on the set variables, considered as parameters. It results of such arguments that all the indeterminates $a_0, \ldots, a_n, b_0, \ldots, b_n$ and λ are shared in two separated subsets, one containing all the variables of $\mathbf{S}_1$ and the other containing all the variables of $\mathbf{S}_2$. Such a decomposition permits to set either a_i, or b_i for $0 \le i \le n$ in such a way that some variables of $\mathbf{S}_1$ and some variables of $\mathbf{S}_2$ remains indeterminate. Then, the irreducibility of $\mathbf{R}$ leads to the fact that either $\mathbf{S}_1$ or $\mathbf{S}_2$ depends only on the variables set to parameters, but this is impossible. In conclusion, the assumption that $\mathbf{S}$ is not irreducible leads to a contradiction. This ends the proof of the lemma.

REFERENCES

[1] E. J. Routh, *Stability of motion*, Edited by A. T. Fuller, Taylor & Francis LTD, London, 1975.

[2] C. Hermite "Sur le nombre de racines d'une équation algébrique comprises entre des limites données", Journal für die Reine und Angewandte Mathematik, vol. 52, pp. 39–51, 1856.

[3] A. Cohn, "Ueber die Anzahl der Wurzelneiner algebraischen Gleichung in einem Kreise", Math. Zeitschrifft, vol. 14, pp. 110–148, 1922.

[4] V. L. Kharitonov, "Generalised stability criterium" (in Russian), Izv. Akad. Nauk. Kazahk., SSR Ser. Fiz. Mat., no 1, pp. 53–57, 1978.

[5] A. C. Bartlett, C. V. Hollot and H. Lin, "Root locations of an entire polytope of polynomials : it suffices to check the edges", Math. Cont. Signals Systems, vol. 1, pp. 61–71, Nov. 1988.

[6] M. Barret and M. Benidir, "On the boundary of the set of Schur polynomials and applications to the stability of 1-D and 2-D digital recursive filters", IEEE Trans. Automatic Control, vol. 39, no. 11, pp. 2335–2339, Nov. 1994.

[7] M. Barret and M. Benidir, "A new algorithm to test the stability of 2-D digital recursive filters", Signal Processing, vol. 37, no. 2, pp. 255-264, May 1994.

[8] F. R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing Company, New-York, vol. 2, ch. 15, 1960.

[9] E. I. Jury, "Robustness of discrete systems : a review", Automation and Remote Control, vol. 10, pp. 571–592, Oct. 1990.

[10] E. Horowitz and S. Sahni, *Fundamentals of Computer Algorithms*, Computer Science Press, Chapter 9,1978.