MULTIRESOLUTION IMAGE DECOMPOSITION WITH COMPLEX STEERABLE PYRAMIDS

G. Jacovitti*, A. Manca*, A. Neri**

* INFOCOM Dpt., University of Rome “La Sapienza”, Via Eudossiana 18, 00814 Rome, Italy
** Electronics Engineering Dept., University of Rome III, Via Vasca navale 84, 00146 Rome, Italy

e-mail: neri@infocom.ing.uniroma1.it

ABSTRACT

In this contribution we present a steerable pyramid based on complex wavelets named *Circular Harmonic Wavelets* (CHW), suited for multiscale feature-based representations. The Circular Harmonic Pyramid (CHP) performs a local windowed Fourier analysis in polar co-ordinates around any point of the image. After a survey on the general properties of the CHP, we illustrate the application of the CHP to the classical problem of image restoration against additive noise.

Keywords: Circular Harmonic Wavelets, Multiscale Image Features Analysis

I. INTRODUCTION

Multiresolution representation is a flexible and computationally effective tool for image processing tasks. In particular, frame-oriented approaches aim to extract significant image attributes at different resolution levels arranged in a pyramidal structure. As an example, the classical laplacian pyramid based on second order derivatives, evaluates zero crossing at different resolutions. Recent advances in the area of the wavelets have provided further contributions to the design of pyramidal image representations. In [1] a pair of wavelets defined by horizontal and vertical smoothed gradient operators is employed for obtaining a hierarchy of edge features, indexed by various scales or resolution levels. In order to capture anysotropic features more explicitly, sets of multiple wavelets individually tuned to angular selective subbands can be employed. In particular in [2] “steerable” wavelets are defined as oriented operators which are rotated copies of each other and whose orientation can be changed by linearly combining the basic ones. Steerable pyramids especially designed for directional n-th order gradient computation at various scales have been proposed in [3].

In this contribution, we present a steerable *Circular Harmonic pyramid* (CHP) based on the *Circular Harmonic Wavelets* (CHW), whose general properties have been recently discussed in [4]. The CHP performs a local windowed Fourier analysis in polar coordinates, around any point of the image, and constitutes a mathematical generalization of the Fourier-Mellin transform.

In this contribution we first review the mathematical properties of the proposed CHP. Then, we illustrate the application of the CHP to the classical problem of image restoration against additive noise. Extending the approach based on first order CHWs [4], we consider the joint Bayesian extraction of multiple features in the CHP. Some examples of image restoration showing the quality improvements with respect to other multiresolution approaches are finally provided.

II. THE CIRCULAR HARMONIC WAVELETS

The CHWs are derived from the class of the so called Circular Harmonic Functions (CHF), also referred in the literature to as Harmonic Angular Filters (HAF). A CHF of order n is a complex-valued, polar separable function $\psi^{(n)} \in L^2(\mathbf{R}^2, d^2\mathbf{x})$ of the form

$$\psi^{(n)}\ [x_1(r,\theta), x_2(r,\theta)] = h(r)\, e^{jn\theta} \tag{1}$$

This class of functions is well known in the domain of optical image processing, where it has been employed for rotation invariant pattern recognition [5],[6].
Following [7], the wavelet analysis of a 2D complex-valued signal (image) f of finite energy, defined on the real plane $\mathbf{R}^2$, can be performed by dilating, rotating and translating a single complex-valued function ψ. In particular, as demonstrated in [4], a CHF $\psi^{(n)}$ of the form (1), defines a *Circular Harmonic Wavelet* of order *n,* iif it satisfies the admissibility condition

$$c_{\psi^{(n)}} = 2\pi \int_0^{\infty} \frac{\left|\tilde{h}_n(\rho)\right|^2}{\rho}\, d\rho < \infty \tag{2}$$

where $\tilde{h}_n(\rho)$ is the n-th order Hankel transform of $h(r)$:

$$\tilde{h}_n(\rho) = 2\pi \int_0^{+\infty} r\, h(r)\, J_n(r\rho)\, dr, \qquad n = 0,1,2,\ldots \tag{3}$$

Then the Integral Wavelet Transform (IWT) of the image f on $L^2(\mathbf{R}^2, d^2\mathbf{x})$ with respect to $\psi^{(n)}$ is the scalar product of f with the transformed wavelet $\psi^{(n)}_{\mathbf{b},\phi,a}$

$$W_\psi f(\mathbf{b},\phi,a) = \langle \psi^{(n)}_{\mathbf{b},f,a} | f \rangle \quad (4)$$

where $\psi^{(n)}_{\mathbf{b},\phi,a} = T_b\, R_\phi\, D_a\, \psi^{(n)}$ is obtained from $\psi^{(n)}$ by applying first the dilation operator D_a leaving invariant the $L^1(\mathbf{R}^2, d^2\mathbf{x})$ norm:

$$D_a: L^1(\mathbf{R}^2, d^2\mathbf{x}) \rightarrow L^1(\mathbf{R}^2, d^2\mathbf{x}), \quad f(\mathbf{x}) = \frac{1}{a^2} f\left(\frac{\mathbf{x}}{a}\right), \quad (5)$$

then the rotation operator R_ϕ :

$$R_\phi: L^2(\mathbf{R}^2, d^2\mathbf{x}) \rightarrow L^2(\mathbf{R}^2, d^2\mathbf{x}), \; f(\mathbf{x}) \rightarrow f(R_{-\phi}\,\mathbf{x}), \quad (6)$$

where R_ϕ denotes the left-action of the rotation group $\boldsymbol{T}$ on the plane

$R_\phi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$,

$$[x_1, x_2] \rightarrow [x_1\cos\phi - x_2 \sin\phi,\; x_1\sin\phi + x_2 \cos\phi] \quad (7)$$

and finally, the translation operator T_b defined as

$$T_b: L^2(\mathbf{R}^2, d^2\mathbf{x}) \rightarrow L^2(\mathbf{R}^2, d^2\mathbf{x}), \quad f(\mathbf{x}) \rightarrow f(\mathbf{x}-\mathbf{b}). \quad (8)$$

From (1) and (4) it follows that the IWT associated to a CHW of n-th order becomes:

$$W_{\psi^{(n)}} \mathrm{f}(\mathbf{b},\phi,\mathrm{a}) =$$
$$= \mathrm{e}^{-\mathrm{j}n\phi} \int_0^\infty \int_0^{2\pi} \frac{\mathrm{r}}{\mathrm{a}^2} \mathrm{h}\left(\frac{\mathrm{r}}{\mathrm{a}}\right) \mathrm{e}^{-\mathrm{j}n\theta} \mathrm{f}\left[\mathrm{r}(\mathbf{x}+\mathbf{b}),\theta(\mathbf{x}+\mathbf{b})\right] \mathrm{dr}\, \mathrm{d}\theta \quad (9)$$

The peculiar form (1) of the CHFs implies that the rotation operator R_ϕ reduces to the multiplication by the complex factor $e^{-jn\phi}$, so that complex CHWs are individually steerable and shiftable. In fact from (9) we directly obtain

$$W_{\psi^{(n)}} \mathrm{f}(\mathbf{b},\phi,\mathrm{a}) = \mathrm{e}^{-jn\phi}\, W_{\psi^{(n)}} \mathrm{f}(\mathbf{b},0,a). \quad (10)$$

In addition, the shiftability in orientation given by (10) implies that the inversion formula simplifies as follows:

$$f(\mathbf{x}) = \frac{2\pi}{c_{\psi^{(n)}}} \int_{\mathbf{R}^+} \frac{da}{a} \int_{\mathbf{R}^2} W_{\psi^{(n)}} f(\mathbf{b},0,a) \psi^{(n)}_{\mathbf{b},0,a}(\mathbf{x}) d^2\mathbf{b} \quad (11)$$

When a CHW $\psi^{(n)}$ satisfies the following *stability condition* for dyadic CHWs

$$0 < A \le \sum_{k=-\infty}^{\infty} \left|\tilde{h}_n(2^{-k}\rho)\right|^2 \le B < \infty \qquad a.e. \quad (12)$$

the samples of its IWT $W_{\psi^{(n)}} f(\boldsymbol{b},\phi,a)$ at scales $a=a_j=2^{-j}$, $j\in$ **Z**, constitute a complete and stable representation of any finite energy image f. In fact, in this case the following inversion formula holds:

$$f(\mathbf{x}) = \sum_m 2^{-2m} \int_{\mathbf{R}^2} W_{\psi^{(n)}} f(\mathbf{b},0,2^{-m}) \psi'^{(n)}_{\mathbf{b},0,2^{-m}}(\mathbf{x}) d^2\mathbf{b} \quad (13)$$

where $\psi'^{(n)}$ is the dyadic dual of $\psi^{(n)}$ defined as:

$$\psi'^{(n)}[x_1(r,\theta), x_2(r,\theta)] = h'(r)\, \mathrm{e}^{jn\theta} \quad (14)$$

and $h'(r)$ is defined in terms of its Hankel transform as follows

$$\tilde{h}'_n(\rho) = \frac{\tilde{h}_n(\rho)}{\sum_k \left|\tilde{h}_n\left(2^{-k}\rho\right)\right|^2} \quad (15)$$

An interesting family of dyadic CHWs is the one generated by CHFs of the form:

$$\chi^{(n)}(r,\theta) = r^n\, e^{-r^2}\, e^{jn\theta} \quad (16)$$

Observe that these CHFs can be thought as generated by differentiation of the zero order Gaussian CHF $\chi(z)$:

$$\chi(z) = e^{-zz^*} = e^{-r^2} \quad (17)$$

where we posed $z= re^{-j\theta}$. This family possesses the remarkable property of being not only angularly but also radially isomorphic with its Fourier spectrum (see [8, pg. 486 n.11.4.29]):

$$\hat{\chi}(\rho,\gamma)^{(n)} = \pi\, 2^{-n} \rho^n e^{-\frac{\rho^2}{4}} (-j)^n e^{jn\gamma}. \quad (18)$$

From this property it can be easily verified that

$$\chi^{(n)}(r,\theta) * \chi^{(m)}(r,\theta) = 2\pi\, 2^{(n+m)/2} \chi^{(n+m)}(2^{-1/2}r,\theta) \quad (19)$$

where $*$ denotes convolution.

In order to derive the basic properties of the CHWs it is worthwhile to interpret (9) in terms of the local Radial Tomographic Projections (RTP), a *windowed local* version of the Radon transform. At this aim we recall that, given an image $f\in L^2(\mathbf{R}^2, d^2\mathbf{x})$, the RTP($\mathbf{b}$,$\theta$) at $\mathbf{b}=(b_1, b_2)$ along the direction θ is defined as

$$RTP(\mathbf{b};\theta) = \int_0^\infty f\left(b_1 + r\cos\theta, b_2 + r\sin\theta\right) w(r) dr \quad (20)$$

where $w(r)$ is a radial weighting function, and r and θ denote polar coordinates with origin in **b**. Since the RTP is periodic, it can be expand into a Fourier series, namely:

$$\mathrm{RTP}(\mathbf{b};\theta) = \sum_n \mathrm{G}_n(\mathbf{b})\, \mathrm{e}^{\,jn\theta} \quad (21)$$

On the other hand, for any point **b** of the real plane $\mathbf{R}^2$, the Fourier coefficients $G_n(\mathbf{b})$ can be computed by taking the scalar product between the image f and the polar separable function

$$\mathrm{s_n}(\mathrm{r},\theta) = \mathrm{g_n}(\mathrm{r})\, \mathrm{e}^{\,jn\theta} \quad (22)$$

where

$$\mathrm{g_n}(\mathrm{r}) = \frac{1}{2\pi} \frac{\mathrm{w(r)}}{\mathrm{r}} \quad (23)$$

so that

$$G_n(\mathbf{b}) = \langle s_n[r(\mathbf{x}-\mathbf{b}), \theta(\mathbf{x}-\mathbf{b})] \,|\, f(\mathbf{x}) \rangle =$$
$$= W_{s_n} f(\mathbf{b},0,1) \quad (24)$$

Thus, for a given scale a, the IWT $W_{s_n} f(\mathbf{b},0,a)$ represents the n-th order harmonic component of a dilated version of f, so that the IWT magnitude reveals the presence of some features, and the phase their orientation. Specifically $W_{s_1} f(\mathbf{b},0,a)$ extracts the first harmonic of the RTP, which is the fundamental harmonic of an edge centered on **x**. In this case, the magnitude of $W_{s_1} f(\mathbf{b},0,a)$ measures the strength of the edge while its phase measures the orientation. Likewise, $W_{s_2} f(\mathbf{b},0,a)$ extracts the fundamental harmonic of line patterns centered on **b**. In this case θ represents the direction of the line. Proceeding in this way, $W_{s_3} f(\mathbf{b},0,a)$ is tuned to trihedric vertices, $W_{s_4} f(\mathbf{b},0,a)$ to orthogonal crosses, and so on. On the other hand the CHFs can be interpreted as smoothed *n*-th order derivatives. This is indicated from the presence of the factor $(-j\rho\, e^{j\gamma})^n$ in eq.(18)

More in general, CHFs of any order may generate CHWs suited for multiscale feature-based representations, that constitute a basis for general steerable wavelets. In particular, CHFs with the same radial profile constitute the angular harmonics of any steerable polar separable wavelet. Thus, a multiscale general feature analysis can be performed by linearly combining the outputs of CHF operators of different scales and orders (CHF filter banks).

III. THE OVERCOMPLETE STEERABLE PYRAMID

The collection of CHWs of different orders constitutes an *overcomplete* dictionary $D = \{\psi^{(n)}_{\mathbf{b},0,a}\}$, so that any image f can be obtained by linearly combining the dictionary elements. The overcomplete nature of the dictionary comes from the fact that CHFs of any order can be represented in terms of a single CHW of an arbitrary order, and therefore the decomposition is not unique. Such a dictionary defines the steerable circular harmonic pyramid (CHP), suited for image processing oriented to the class of the above said features. In fact the CHP provides local representation of an image in terms of harmonic components of the RTP around generic point. Thus, the CHP combines multiscale decomposition with tomographic decomposition, while possessing the steerability property.

A possible representation of any finite energy image f based on a dyadic CHP is:

$$\mathrm{f}(\mathbf{x}) = \sum_{n}\sum_{m} 2^{-2m}\int_{\mathbf{R}^2} W_{\psi^{(n)}}\, \mathrm{f}(\mathbf{b},0,2^{-m})\,\psi''^{(n)}_{\mathbf{b},0,2^{-m}}(\mathbf{x})\,d^2\mathbf{b} \tag{25}$$

where $\psi''^{(n)}$ is

$$\psi''^{(n)}[x_1(r,\theta), x_2(r,\theta)] = h''_n(r)\, e^{jn\theta} \tag{26}$$

and $h''_n(r)$ is defined in terms of its Hankel transform as follows

$$\tilde{h}''_n(\rho) = \frac{\tilde{h}_n(\rho)}{\sum_n \sum_k \left|\tilde{h}_n\left(2^{-k}\rho\right)\right|^2} \tag{27}$$

In general the CHP can be employed in a wide variety of image processing applications, including edges and vertices detection, motion estimation, image fusion and enhancement, texture classification, etc.

Here we briefly describe an application of Bayesian restoration of noisy images.

Denoting with $\mathbf{z}^M(n,\mathbf{b},m)$ the 2-D real array associated to the complex wavelet coefficient $W_{\psi^{(n)}} f^M(\mathbf{b},0,2^{-m})$ of the noisy image f^M, we write $\mathbf{z}^M(n,\mathbf{b},m)$ as the sum of the wavelet coefficient of the original image $\mathbf{z}(n,\mathbf{b},m)$ and a complex Gaussian zero mean noise $\Delta\mathbf{z}(n,\mathbf{b},m)$ with covariance matrix $\mathbf{R}_N^{(n,m)}$. In addition we model the wavelet coefficients of the original image $\mathbf{z}(n,\mathbf{b},m)$ as random variables with probability density function given by a complex Gaussian mixture, i.e. a weighted sum of Gaussian distributions:

$$p_{\mathbf{Z}}[\mathbf{z}(n,\mathbf{b},m)] = \sum_{i=1}^{M} \lambda_i\, G_2\left[\mathbf{z}(n,\mathbf{b},m);0,\mathbf{R}_{F_i}^{(n,m)})\right], \tag{28}$$

where $G_2[\mathbf{z};\mu,\mathbf{R}]$ is the Gaussian p.d.f. of a complex r.v. **z** with expectation μ and covariance matrix **R**.

Then the evaluation of the conditional expectation $\hat{\mathbf{z}}(n,\mathbf{b},m)$ of $\mathbf{z}(n,\mathbf{b},m)$ yields:

$$\hat{\mathbf{z}}(n,\mathbf{b},m) = $$

$$= \sum_i w_i\left[\mathbf{z}(n,\mathbf{b},m)\right]\mathbf{R}_{F_i}^{(n,m)}\left(\mathbf{R}_N^{(n,m)} + \mathbf{R}_{F_i}^{(n,m)}\right)^{-1}\mathbf{z}(n,\mathbf{b},m) \tag{29}$$

where the weights $w_i[\mathbf{z}(n,\mathbf{b},m)]$ are given by:

$$w_i\left[\mathbf{z}(n,\mathbf{b},m)\right] = \frac{\lambda_i\, G_2\left[z(n,\mathbf{b},m),0,\mathbf{R}_N^{(n,m)} + \mathbf{R}_{F_i}^{(n,m)}\right]}{\sum_i \lambda_i G_2\left[z(n,\mathbf{b},m),0,\mathbf{R}_N^{(n,m)} + \mathbf{R}_{F_i}^{(n,m)}\right]}. \tag{30}$$

Applying such an estimate to the individual components of the CHP and recovering the images with the inversion formula (25) constitutes an effective noise suppression procedure similar to the shrinking method based on a minimax criterion in a framework of multiresolution restoration [9].

The associated reconstruction scheme is shown in Fig1.

An example of such an application is shown in Fig. 2. The original image (fig. 2a) is degraded with an additive Gaussian noise (fig. 2b). The noisy image is decomposed with a CHP pyramid using the dyadic CHWs given by (16), from the first to the fourth order at four resolution levels. Finally in fig. 2c the result of the restoration process is shown.

More sophisticated denoising schemes could be implemented starting from a matching pursuit based CHP decomposition, inspired to [10].

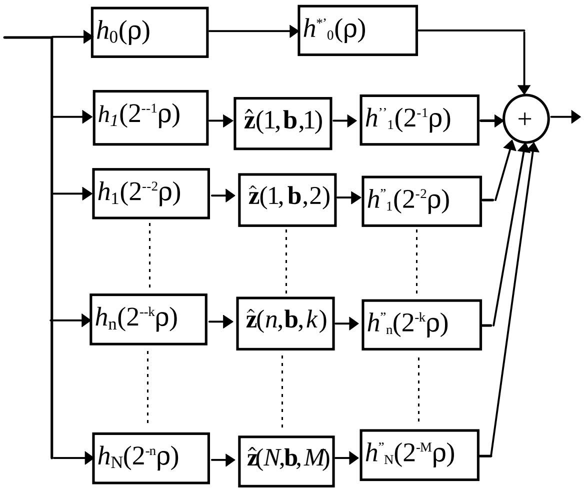

Fig. 1. Restoration block diagram for *N* CHW order and *M* scales.

REFERENCES

[1] S. Mallat and S. Zhong, "Characterisation of Signals from Multiscale Edges", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 14, no.7, pp. 710-732, 1992.

[2] E.P. Simoncelli, W.T. Freeman, E. H. Adelson, D.J. Heeger, "Shiftable Multiscale Transforms", *IEEE Trans. on Information Theory*, Vol. 38, no. 2, pp. 587- 607, March 1992.

[3] E.P. Simoncelli, W.T. Freeman, "The Steerable Pyramid: A Flexible Architecture for Multi-Scale Derivative Computation", *IEEE Proc. of ICIP-95*, Vol. III, pp. 444-447, Oct. 1995.

[4] G. Jacovitti, A.Neri, "Multiscale image Features Analysis with Circular Harmonic Wavelets", in *Wavelets Appl. in Signal and Image Processing III*, Proc. SPIE's 2569,pp. 363-372, July 1995.

[5] W. L. Anderson, "Application of Optical Fourier Transforms", H. Stark Editor, Chapter 3, Academic Press, 1982, pp. 89-130.

[6] J. Rosen, J. Shamir, "Circular Harmonic Phase Filters for Efficient Rotation - Invariant Pattern Recognition", *Applied Optics*, Vol.27, no.14, pp.2895-2899, July 1988.

[7] M. Holschneider, *Wavelets: An Analysis Tool*, Oxford Science Pub., Clarendon Press, Oxford 1995.

[8] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc., New York, 1972.

[9] D.L. Donoho, "De-Noising via Soft-Thresholding", *Tech. Rep.*, Statistics, Stanford, 1992.

[10] F. Bergeaud, S. Mallat, "Processing images and sounds with matching pursuits", in *Wavelets Application in Signal and Image Processing III*, Proc. SPIE's 2569, pp. 2-13, July 1995.

Fig2.a. Original image.

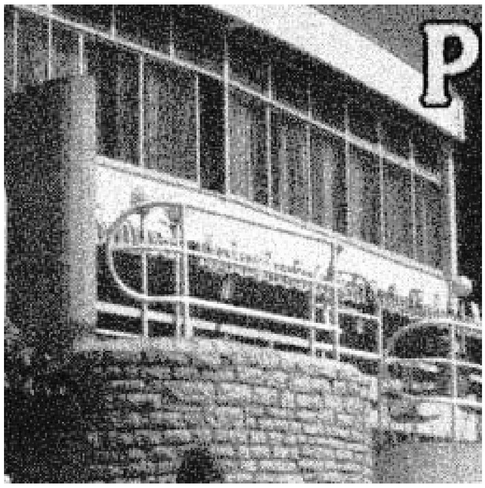

Fig2.b. Noisy version (additive Gaussian noise, SNR = 16 dB).

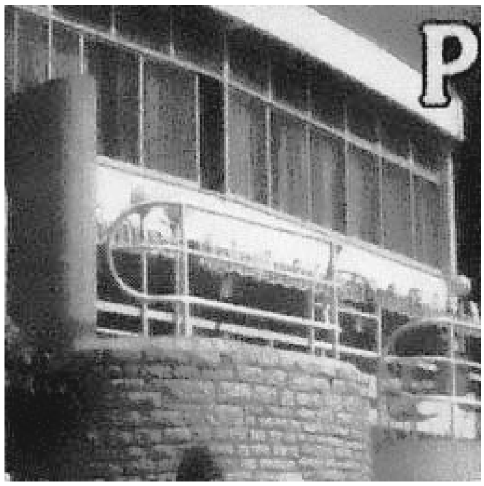

Fig2.c. Restored image.