Parameter estimation of exponentially damped sinusoids using second order statistics

K. Abed-Meraim*, A. Belouchrani**, A. Mansour***, and Y. Hua*

* Department of Electrical and Electronics Engineering,
The University of Melbourne, Parkville, Victoria 3052, Australia, a.karim@ee.mu.OZ.AU
** Department of Electrical Engineering and Computer Sciences,
The University of California, Berkeley CA 94720, U.S.A, adel@robotics.eecs.berkeley.edu
*** LTIRF - INPG, 46 Av. Felix Viallet, 38031 Grenoble, mansour@tirf.inpg.fr

Abstract

In this contribution, we present a new approach for the estimation of the parameters of exponentially damped sinusoids based on the second order statistics of the observations. The method may be seen as an extension of the minimum norm principal eigenvectors method (see [1]) to cyclo-correlation statistics domain. The proposed method exploits the nullity property of the cyclo-correlation of stationary processes at non-zero cyclo-frequencies [2]. This property allows in a pre-processing step to get rid from stationary additive noise. This approach presents many advantages in comparison with existing higher order statistics based approaches [3]: **(i)** *First it deals only with second order statistics which require generally few samples in contrast to higher-order methods,* **(ii)** *it deals either with Gaussian and non-Gaussian additive noise, and* **(iii)** *also deals either with white or temporally colored (with unknown autocorrelation sequence) additive noise. The effectiveness of the proposed method is illustrated by some numerical simulations.*

1. Introduction

Parameter estimation of exponentially damped sinusoids from a finite subset of noisy observations is a very common problem in signal processing. Such a problem arises in many practical fields and has already received considerable attention in the signal processing literature [1, 3, 4, 5, 6]. For additive white Gaussian noise, the damped sinusoids parameters can be estimated using the iterative quadratic maximum likelihood method (IQML) [6]. Prony's method [1] and matrix pencil (MP) [5] method can be applied when the additive noise contribution can be neglected. Higher order statistics based methods can be used in the case of Gaussian additive noise [3, 7]. Others estimation approaches use an autoregressive modeling of the additive colored noise [8, 9]. Our method can be applied for any stationary[1] additive noise process. The method is based on the use of the cyclo-correlation function of the observed signal and will be referred as CCEM (Cyclo-Correlation based Estimation Method). The main motivation behind the use of cyclo-correlation statistics in this problem lies in their ability to suppress noise under stationarity hypothesis.

[1]Even this assumption can be relaxed at the price of more effort and notation.

2. The second order statistics based method

Let $(y(n))_{n\in\mathbb{Z}}$ be a scalar observed signal modeled for any instant $n \geq 0$ as L exponentially damped complex sinusoids corrupted by additive noise:

$$y(n) = \sum_{m=1}^{L} h_m e^{b_m n} + w(n), \quad n = 0, 1, \cdots \tag{1}$$

where the complex constants are defined as

$$h_m = a_m e^{j\theta_m}, \quad b_m = -\alpha_m + jf_m, \quad \textit{with} \quad \alpha_m > 0 \tag{2}$$

and $w(n)$ denotes the additive noise which is assumed here to be a stationary random process. Note that the a_m and θ_m are respectively the amplitude and the initial phase of the m-th signal; its damping and frequency factors are respectively α_m and f_m. The problem addressed here deals with estimation of the frequencies $\{f_m\}$, damping factors $\{\alpha_m\}$, and when desired, complex amplitudes $\{h_m\}$ from a finite amount of observed data $y(n), n = 0, .., N-1$. In the sequel, we first give the explicit expression of the observed signal cyclo-correlation and then we show how one can estimate both of the damping and the frequency factors using a linear prediction approach.

2.1. Cyclo-correlations of exponential signals

Consider the noiseless signal in (1). Let $\beta \neq 0$ be the considered cyclo-frequency and let $r^\beta(k)$ denotes the k-th cyclo-correlation factor at the cyclo-frequency β.

$$r^\beta(k) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} y(n+k)y(n)^* e^{j\beta n} \tag{3}$$

$$
\begin{aligned}
r^{\beta}(k) &= \sum_{m,l=1}^{L} h_m h_l^* e^{k b_m} \sum_{n=0}^{\infty} e^{(b_m + b_l^* + j\beta) n} \\
&= \sum_{m=1}^{L} A^{\beta}(m) e^{k b_m} \qquad (4)
\end{aligned}
$$

where

$$
A^{\beta}(m) = \sum_{l=1}^{L} \frac{h_m h_l^*}{1 - e^{b_m + b_l^* + j\beta}} \qquad (5)
$$

¿From (4), the theoretical cyclo-correlation of exponential signals may be seen as yet another exponential signal with the same pole location but with different amplitudes and initial phases.

In practice, we have only a finite data length (N observations). In this case, the cyclo-correlation coefficients are estimated by

$$
\hat{r}^{\beta}(k) = \sum_{n=n_0}^{n_1} y(n+k) y(n)^* e^{j\beta n}
$$

where $n_0 = max(0, -k)$ and $n_1 = min(N-1, N-1-k)$.

The main advantage, in dealing with cyclo-correlation instead of correlation function, is that noise contribution is considerably reduced in the former case. In fact, due to the stationarity assumption of the noise process, we have:

$$
\frac{1}{N} \sum_{n=n_0}^{n_1} w(n+k) w(n)^* e^{j\beta n} \stackrel{N \to \infty}{\longrightarrow} 0
$$

when

$$
\frac{1}{N} \sum_{n=n_0}^{n_1} w(n+k) w(n)^* \stackrel{N \to \infty}{\longrightarrow} \rho(k)
$$

$\rho(k)$ being the k-th correlation factor of the noise process. Generally, for additive colored noise, the signal to noise ratio (SNR) gain, can be considerable since, for $\rho(k) \neq 0$, we have

$$
\left| \frac{\sum_{n=n_0}^{n_1} w(n+k) w(n)^*}{\sum_{n=n_0}^{n_1} w(n+k) w(n)^* e^{j\beta n}} \right| \longrightarrow \infty
$$

2.2. A linear prediction approach

It is well known that for the signal $r^{\beta}(k)$ there exists a unique set of complex coefficients $\{\mathrm{h}_i, i = 0, \cdots, L\}$ with $\mathrm{h}_0 = 1$ such that [10]

$$
\mathrm{h}_0 r^{\beta}(k) + \mathrm{h}_1 r^{\beta}(k-1) + \cdots + \mathrm{h}_L r^{\beta}(k-L) = 0 \qquad (6)
$$

where the polynomial $\mathrm{h}(z) = \mathrm{h}_0 z^L + \mathrm{h}_1 z^{L-1} + \cdots + \mathrm{h}_L$ is the linear prediction (LP) polynomial for the noiseless signal r^{β}, and has roots $z_i = e^{b_i}$, $1 \leq i \leq L$. Thus, if the coefficient vector $\mathrm{h} = [\mathrm{h}_0, \cdots, \mathrm{h}_L]^T$ is estimated by some identification method, rooting of $\mathrm{h}(z)$ will provide estimates of b_i, $1 \leq i \leq L$.

Equation (6) can be manipulated for M lags ($k = k_0, \cdots, M + k_0 - 1$) into the following vectorial form

$$
H r^{\beta} = R^{\beta} \mathrm{h} = 0 \qquad (7)
$$

$$
H = \begin{pmatrix} \mathrm{h}_L & \cdots & \mathrm{h}_0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & \cdots & \mathrm{h}_L & \cdots & \mathrm{h}_0 \end{pmatrix}
$$

and

$$
R^{\beta} = \begin{pmatrix} r^{\beta}(k_0) & \cdots & r^{\beta}(k_0 - L) \\ \vdots & & \vdots \\ r^{\beta}(k_0 + M - 1 - L) & \cdots & r^{\beta}(k_0 + M - 2L) \end{pmatrix}
$$

Remarks: 1) The minimum number of independent equations required to estimate the L unknown coefficients of the LP polynomial is L. In other words, the number of lags should be chosen such that $M \geq L$.

2) In equation (7), we assume implicitly that the number of damped sinusoids is known. In practice, it is not the case and the sinusoids number needs to be estimated. Many estimation procedures can be found in the literature [11, 12, 13], however this problem is beyond the scoop of this paper and will not be treated in the sequel.

3) Since the number of terms involved in the estimation of the k-th cyclo-correlation factors is a decreasing function of $|k|$, the value of k_0 should be chosen such that $\max(|k_0 - L|, |k_0 - L + M - 1|)$ is minimum.

4) Due to the finite sample length and the noise effects, equation (7) has not an exact solution. In practice, the LP polynomial h can be estimated by minimizing the least squares criterion:

$$
\hat{\mathrm{h}} = \underset{\mathrm{h}}{\mathrm{Argmin}} \, \|R^{\beta}\mathrm{h}\|^2 = \mathrm{h}^* R^{\beta *} R^{\beta} \mathrm{h} \qquad (8)
$$

5) The linear prediction approach is not the unique choice to estimate the damped sinusoids parameter. Others estimation techniques [5, 1] can be applied to the preprocessed signal r^{β}. In particular, the Cyclo-Correlation based Matrix Pencil method (referred as CCMP method) will be used in the sequel for performance comparison (see section 3).

2.3. Variations on the criterion

Equations (7) and (8) provide a criterion to estimate the coefficients of the LP polynomial which characterizes uniquely the damped sinusoids to be estimated. Other interesting strategies for the estimation procedure may be considered that will not be detailed here, due to the lack of space. These include:

- **Using a weighting matrix** in the criterion (8), in order to improve the estimation performance. Therefore, the LP polynomial h can be estimated by minimizing the weighted least squares criterion:

  $$
  \hat{\mathrm{h}} = \underset{\mathrm{h}}{\mathrm{Argmin}} \, \|R^{\beta}\mathrm{h}\|_W^2 = \mathrm{h}^* R^{\beta *} W R^{\beta} \mathrm{h} \qquad (9)
  $$

  where W is any positive definite weighting matrix. In particular, it can be noticed that for $W = I$ (resp. for $W = (HH^*)^{-1}$) we retrieve the Prony (resp. the IQML) criterion [6] applied to the cyclo-correlation sequence. An optimal choice of the weighting matrix can be provided based on a statistical analysis of the estimation error. This study will be detailed in a forthcoming paper.

is, to increase the signal to noise ration (SNR). Using (5), it is easy to see that the Riemann integral of the cyclo-correlation coefficients is given by:

$$\int_{\beta_0}^{\beta_1} r^{\beta}(k)d\beta = \sum_{m=1}^{L} B(m)e^{b_m k} \qquad (10)$$

where

$$B(m) = \sum_{l=1}^{L} h_m h_l^* [j(\log(1 - e^{b_m + b_l^* + j\beta_1}) - \log(1 - e^{b_m + b_l^* + j\beta_0})) + \beta_1 - \beta_0] \qquad (11)$$

Equation (10) can take the general form:

$$\int_{\beta_0}^{\beta_1} f(\beta) r^{\beta}(k) d\beta$$

where $f(\beta)$ is an appropriate weight ing function. β_0, β_1 and $f(\beta)$ should be chosen to maximize the amplitude coefficients $|B(m)|, m = 1, \cdots L$. Of course, such maximization procedure is highly non-linear, and in practice some approximation or sub-optimal schemes should be rather considered.

- **Using different cyclo-correlation coefficients.** For example, if the additive noise is complex circular (which implies in particular that $E(w(n+k)w(n)) = 0$), we can exploit the circularity of the additive noise, by replacing (3) by

$$c^{\beta}(k) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} y(n+k)y(n)e^{j\beta n}$$

  In this case, we can reduce the noise contribution thanks to both of the cyclo-stationarity and circularity effects.
- **Using an iterative estimation procedure**, which is in fact necessary in the case where optimal (or sub-optimal) choices of the weighting matrix W, the cyclo-frequencies (β_0, β_1), and the weighting function $f(\beta)$ are considered. Such an optimal choices should depend on the unknown parameters, and at least a two step estimation procedure is necessary to (i) first estimate the damped sinusoids using the least square criterion (8), then (ii) estimate the optimization parameters function of the previous data model parameters.

3. Simulation results

We present here some numerical simulations to assess the performance of our algorithm. The simulation corresponds to the example described in [3]. The data model is given by

$$y(n) = e^{b_1 n} + e^{b_2 n} + w(n)$$

where $b_1 = -0.2 + j(0.42)2\pi$ and $b_2 = -0.1 + j(0.52)2\pi$. For each experiment, the sample size is set to $N = 64$ and $N_r = 100$ independent Monte-Carlo simulations are performed. The performance is measured by the mean-square error (MSE) defined by

$$MSE = \sqrt{\frac{1}{N_r}\sum_{r=1}^{N_r} \|\hat{b}_r - b\|^2}$$

Figure 1 (respectively figure 2) compares the performances of our method with those of the MP, CCMP, and IQML methods for white Gaussian additive noise, in the case where $K = 16$ realizations (respectively $K = 1$ realization) are available. We chose $M = 18$, $W = (HH^*)^{-1}$, $\beta_0 = 0.1$, $\beta_1 = 1$ and $f(\beta) = 1$. The plots show the MSE (in dB) as a function of the SNR in dB (the SNR is defined by SNR$= 1/\sigma^2$, where σ^2 is the additive noise power). This show the high performance and robustness to additive noise of the proposed method.

Figure 3 (respectively figure 4) compares the performances of our method with those of the MP, CCMP, and IQML methods for non-Gaussian colored additive noise, in the case where $K = 16$ realizations (respectively $K = 1$ realization) are available. The noise signal is generated by filtering a complex circular uniform distributed white process by an MA (Moving Average) model of order two given by

$$h(z) = 1 + 0.7z^{-1} + 0.49z^{-2}$$

As for the first experiment, We chose $M = 18$, $W = (HH^*)^{-1}$, $\beta_0 = 0.1$, $\beta_1 = 1$ and $f(\beta) = 1$. The plots show the MSE (in dB) as a function of the SNR (in dB). As we can see, our method offers a significant gain of the estimation performance.

It is worth to notice that the simulation results shown in this section depend on the choice of the cyclo-frequency β. More detailed studies still are necessary to assess the effect of the cyclo-frequency choice on the estimation performance and to verify whether the good estimation behavior depends highly or weakly on the considered cyclo-frequency.

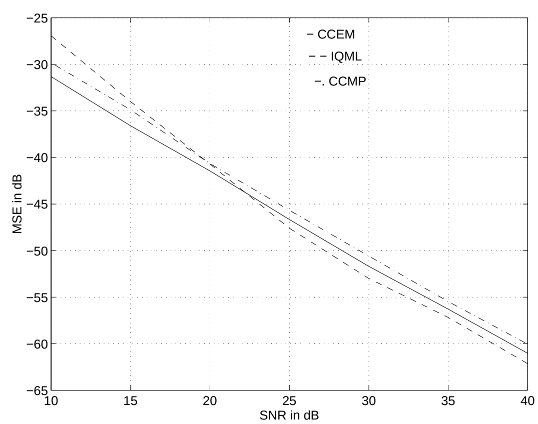

Figure 1

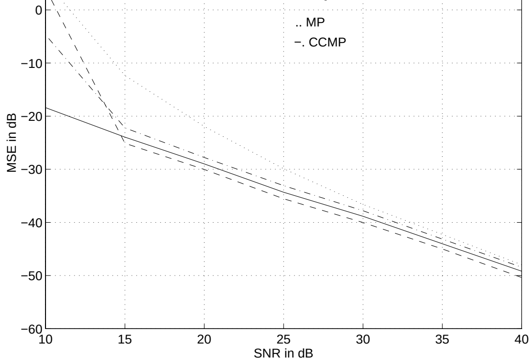

Figure 2

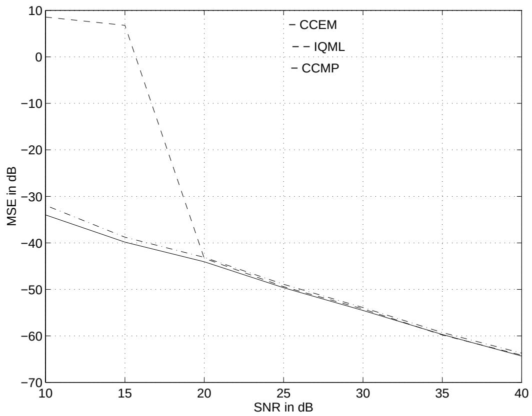

Figure 3

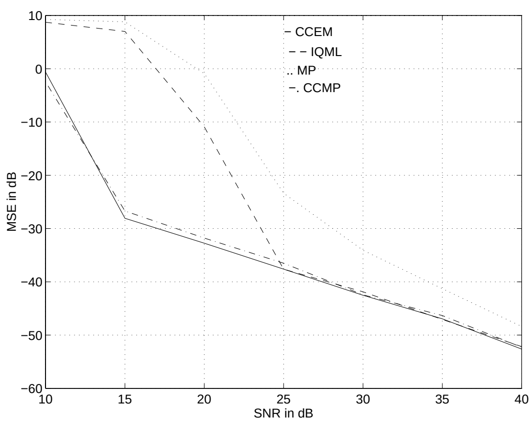

Figure 4

4. Conclusion

Second order cyclo-stationary statistics were used to derive a new approach for the estimation of the parameters of exponentially damped sinusoids. The signal parameters were calculated by polynomial rooting of a vector of coefficients, which was the solution of a linear system of equations involving cyclo-correlation coefficients. The main advantage very weak assumptions or *a priori* knowledge on the noise distribution.

For the evaluation of the performance of the new method, the IQML algorithm was used for comparison. It was demonstrated through simulations that when the additive noise is non-Gaussian or colored with unknown autocorrelation function, the proposed method offers a significant improvement in the estimation performance. Furthermore, even in the case of additive white Gaussian noise our method seems to be more robust to additive noise especially for very low SNR.

References

[1] R. Kumaresan and D.W. Tufts, "Estimating the paramaters of exponentially damped sinusoids and pole-zero modeling in noise," *IEEE Tr. on Acoust. Speech and Sig. Proc.*, vol. 30, pp. 833–840, Dec. 1982.

[2] W. Gardner, "Exploitation of the spectral redundancy in cyclostationary signals," *Signal Processing Mag.*, vol. 8, no. 2, pp. 14–37, 1991.

[3] C.K. Papadopoulos and C.L. Nikias, "Parameter estimation of exponentially dumped sinusoids using higher order statistics," in *IEEE Tr. on Sig. Proc.*, vol. 38, pp. 1424–1435, Aug. 1990.

[4] S. Kay, *Modern spectral estimation.* Englwood Cliffs, NJ: Prentice-Hall, 1988.

[5] Y.Hua and T.K.Sarkar, "Matrix pencil method for estimating paramaters of exponentially damped / undamped sinusoids in noise," *IEEE Tr. on Acoust. Speech and Sig. Proc.*, vol. 38, pp. 814–824, May 1990.

[6] Y.Bresler and A. Macovski, "Exact maximul likelihood parameter estimation of superimposed exponential signals in noise," *IEEE Tr. on Acoust. Speech and Sig. Proc.*, vol. 34, pp. 1081–1089, 1986.

[7] D.P. Ruiz, M.C. Carrion, A. Gallego, and A. Medouri, "Parameter estimation of exponentially damped sinusoids using higher order correlation-based approach," *IEEE Tr. on Sig. Proc.*, vol. 43, pp. 2665–2677, Nov. 1995.

[8] E.H. Satorius and J.R. Zeidler, "Maximum entropy spectral analysis of multiple sinusoids in noise," *Geophysics*, vol. 43, pp. 1111–1118, Oct. 1978.

[9] A.H.Nehorai and M.Morf, "Enhancement of sinusoids in colored noise and the weightning performance of exact least squares predictors," *IEEE Tr. on Acoust. Speech and Sig. Proc.*, vol. 30, pp. 353–362, June 1982.

[10] S.L. Marple, *Jr. Digital spectral analysis with applications.* Englewood Cliffs, NJ: Prentice Hall, 1987.

[11] K. Konstantinides and K. Yao, "Statistical analysis of effective singular values in matrix rank determination," *ieeesp*, vol. 36, pp. 757–763, May 1988.

[12] M. Wax and T. Kailath, "Detection of signals by information theoretic criteria," *IEEE Tr. on Sig. Proc.*, vol. 33, pp. 387–392, Apr. 1985.

[13] V.U.Reddy and L.S. Biradar, "SVD-based information theoretic criteria for detection of the number of damped/undamped sinusoids and their performance analysis," *IEEE Tr. on Sig. Proc.*, vol. 41, pp. 2872–2881, Sept. 1993.