SUBSPACE-BASED PARAMETER ESTIMATION OF SYMMETRIC NON-CAUSAL AUTOREGRESSIVE SIGNALS FROM NOISY MEASUREMENTS*

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ABSTRACT

The notion of Symmetric Non-causal Auto-Regressive Signals (SNARS) arises in several, mostly spatial, signal processing applications. In this paper we introduce a subspace fitting approach for parameter estimation of SNARS from noise-corrupted measurements. We show that the subspaces associated with a Hankel matrix built from the data covariances contain enough information to determine the signal parameters in a consistent manner. Based on this result we propose a MUSIC (MUltiple SIgnal Classification)-like methodology for parameter estimation of SNARS. Compared with the methods previously proposed for SNARS parameter estimation, our SNARS-MUSIC approach is expected to possess a better trade-off between computational and statistical performances.

1 INTRODUCTION

SNARS applications include image reconstruction or deblurring, astronomical and seismic data processing, time series interpolation and spectral estimation, among others; for details see [1, 2] and the references therein. Motivated by the practical relevance of SNARS, several papers in the recent signal processing literature have addressed the problem of estimating the parameters of this type of signals. The paper [2] proposed a Prediction Error Minimization (PEM) approach to SNARS parameter estimation. In the class of estimation methods based on second order statistics, the PEM is generally the most accurate one. However, the PEM requires a multidimensional search over the parameter space, a task for which there is currently no computationally convenient and reliable algorithm. Owing to the aforementioned drawback of PEM, the recent paper [1] made use of the Yule-Walker (YW) approach to derive a computationally simpler SNARS parameter estimation method. However, the computational simplicity of the YW-based method of [1] is obtained at the price of a degraded statistical performance. In particular, unlike PEM, the latter approach violates the parsimony principle by estimating the parameters in a redundant signal model.

In this paper we make use of a subspace-based approach to derive a MUSIC-like methodology for parameter estimation of SNARS. Compared with the aforementioned approaches, the SNARS-MUSIC has a number of advantages. First, it makes a better compromise between the computational and the statistical performances. More exactly, SNARS-MUSIC is usually much simpler as well as more reliable than PEM, at the price of only a slightly degraded statistical accuracy. Secondly, unlike YW and PEM approaches, the SNARS-MUSIC has a naturally associated procedure for order estimation, as explained in the sections to follow.

2 SIGNAL MODEL

By definition, a discrete-time SNARS satisfies the following equation:

$$B(z, z^{-1})x(t) = \varepsilon(t), \qquad t = 0, \pm 1, \pm 2, \ldots \tag{1}$$

where z^{-1} and z are the unit delay and advance operators, respectively (i.e., $z^{-k}x(t) = x(t-k)$ for $k = 0, \pm 1, \pm 2, \ldots$), $\{\varepsilon(t)\}$ is a zero mean white noise sequence with unit variance, and

$$B(z, z^{-1}) = b_n z^{-n} + \cdots + b_1 z^{-1} + b_0 + b_1 z + \cdots + b_n z^n. \tag{2}$$

Assume that we observe a noise-corrupted version of $x(t)$,

$$y(t) = x(t) + w(t), \qquad t = 1, 2, \ldots \tag{3}$$

where $\{w(t)\}$ is a white noise sequence with zero mean and variance denoted by σ_w^2. We assume that $w(t)$ and $\varepsilon(t)$ are uncorrelated with one another,

$$E[w(t)\varepsilon(s)] = 0, \qquad \text{for all } t, s. \tag{4}$$

Hereafter, the symbol E stands for the statistical expectation operator. We also assume that the zeros of $B(z, z^{-1})$ in (2) are strictly bounded away from the unit circle. Under these assumptions, $y(t)$ in (3) is a stationary (non-causal) signal that possesses a power spectral density (PSD) given by

$$\Phi_y(z, z^{-1}) = \frac{1}{B^2(z, z^{-1})} + \sigma_w^2. \tag{5}$$

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Furthermore, by the spectral factorization theorem (see, e.g., [3]), the polynomial $B(z,z^{-1})$ can be written as

$$B(z,z^{-1}) = D(z)D(z^{-1})/\sigma \tag{6}$$

where

$$D(z) \triangleq 1 + d_1 z + \cdots + d_n z^n \neq 0 \quad \text{for } |z| \leq 1. \tag{7}$$

Inserting (6) into (5), we obtain

$$\Phi_y(z,z^{-1}) = \frac{\sigma^2 + \sigma_w^2 D^2(z)D^2(z^{-1})}{D^2(z)D^2(z^{-1})}. \tag{8}$$

By making use of the spectral factorization theorem, once again, we can write the numerator in (8) as follows:

$$\sigma^2 + \sigma_w^2 D^2(z)D^2(z^{-1}) = \sigma_v^2 C(z)C(z^{-1}) \tag{9}$$

where

$$C(z) \triangleq 1 + c_1 z + \cdots + c_{\overline{n}} z^{\overline{n}} \neq 0 \quad \text{for } |z| \leq 1 \tag{10}$$

and where $\overline{n} \triangleq 2n$. It follows from (8) and (9) that the SNARS-plus-white-noise model, (1)–(3), is *spectrally equivalent* to the following "standard" $\overline{n}$th-order autoregressive moving average (ARMA) signal model:

$$D^2(z^{-1})y(t) = C(z^{-1})v(t) \tag{11}$$

where $\{v(t)\}$ is a zero mean white noise sequence with variance equal to σ_v^2.

The problem of interest herein can now be stated as follows: estimate the SNARS parameters $\{b_k\}_{k=0}^n$, or essentially equivalent: estimate $\{d_k\}_{k=1}^n$ and σ^2, from a sample of N noise-corrupted measurements $\{y(t)\}_{t=1}^N$. We will also briefly address the problem of estimating the signal order n. To solve these estimation problems we make use of the ARMA model (11). The idea to employ the spectrally equivalent signal model (11) to solve the original parameter estimation problem was apparently used for the first time in [2]. However, the parameter estimation methods devised in [2] are computationally complex. A computationally much simpler approach, which is also based on the ARMA model (11), has recently been proposed in [1]. However, the method in these references estimates the parameters in a non-parsimonious manner (basically, it estimates the coefficients of $D^2(z)$ instead of estimating the coefficients of $D(z)$), and hence it can be expected to have a degraded statistical performance. The subspace based MUSIC-like approach of this paper estimates the coefficients of $D(z)$ and σ^2 directly. Additionally, the SNARS-MUSIC, being a subspace fitting approach, enjoys the excellent statistical accuracy of this class of parameter estimation methods (see, e.g., [4, 5]). In what concerns the computational burden, SNARS-MUSIC requires a search over a two-dimensional space, which can be organized in an efficient manner (as described in [6]).

3 SNARS HANKEL COVARIANCE MATRIX PROPERTIES

In this section we present a subspace property of the Hankel matrix built from the covariances of a SNARS. The SNARS-MUSIC parameter estimator, to be presented in the next section, is obtained in a straightforward manner from this property. Let

$$r(k) = E[y(t)y(t-k)] \tag{12}$$

and define (for $m, \overline{m} > n$)

$$R \triangleq \begin{pmatrix} r(1) & r(2) & \cdots & r(\overline{m}) \\ r(2) & r(3) & \cdots & r(\overline{m}+1) \\ \vdots & \vdots & & \vdots \\ r(m) & r(m+1) & \cdots & r(m+\overline{m}-1) \end{pmatrix}. \tag{13}$$

Theorem 3.1 *The Hankel covariance matrix R, associated with an nth-order SNARS, can be factorized as*

$$R = \Gamma\Omega^T. \tag{14}$$

Let

$$\begin{pmatrix} f(\lambda) & f'(\lambda) \end{pmatrix} \triangleq \begin{pmatrix} 1 & 0 \\ \lambda & 1 \\ \lambda^2 & 2\lambda \\ \vdots & \vdots \\ \lambda^{m-1} & (m-1)\lambda^{m-2} \end{pmatrix}. \tag{15}$$

Then Γ is given by

$$\Gamma = (f(\lambda_1)\, f'(\lambda_1) \cdots f(\lambda_n)\, f'(\lambda_n)) \tag{16}$$

(the expression of Ω is not important for the present discussion, see [6]). Furthermore, under the assumption that m and $\overline{m}$ are chosen larger than $\overline{n}$, both Γ and Ω have rank equal to $\overline{n}$. Hence

$$\text{rank}\,(R) = \overline{n} \tag{17}$$

and the columns of Γ span the range space of R.

Proof: see [6].

The rank property (17) has a clear potential for SNARS order estimation. Let $\widehat{R}$ be an estimate of R, computed from the available sample of N data points by replacing $r(k)$ (for $k = 1, \ldots, m+\overline{m}-1$) in (13) by

$$\hat{r}(k) \triangleq \frac{1}{N} \sum_{t=k+1}^{N} y(t)y(t-k). \tag{18}$$

We can then estimate the rank of R from $\widehat{R}$ by using the "rule T" recently introduced in [7]. The reader is referred to the cited reference for the details of the rank/order estimation scheme.

The results of Theorem 3.1 also have a potential for SNARS parameter estimation, as described in the next sections.

4 ESTIMATION OF THE COEFFICIENTS OF D

Clearly, the estimation of $\{d_k\}_{k=1}^n$ can be reduced to the estimation of $\{\lambda_p\}_{p=1}^n$, the roots $z^n D(z^{-1})$. By using the fact that rank $(R) = \overline{n}$, we can write the singular value decomposition of R as

$$R = (\underbrace{S}_{\overline{n}} \quad \underbrace{G}_{m-\overline{n}}) \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \begin{matrix} \}\overline{n} \\ \}m-\overline{n} \end{matrix} \quad (19)$$

In the above equation the matrices $(S \ G)$ and $(V_1 \ V_2)$ are orthogonal, and Σ is an $\overline{n} \times \overline{n}$ diagonal and nonsingular matrix. Since the columns of S, like those of Γ, span the range space of R, it readily follows that

$$G^T \Gamma = 0. \quad (20)$$

From (20) and in view of (16),

$$f^*(\lambda) G G^T f(\lambda) + f'^*(\lambda) G G^T f'(\lambda) = 0 \quad (21)$$

for $\lambda = \lambda_1, \ldots, \lambda_n$, where the superscript "$*$" denotes the conjugate transpose, and where use was made of the fact that G is real-valued. Hence, the parameters of interest $\{\lambda_p\}_{p=1}^n$ are the solutions of the equation in (21). Furthermore, it can be shown that they are the only solutions [6].

The previous observations can be exploited to estimate $\{\lambda_p\}_{p=1}^n$ in the following way. Let $\widehat{G}$ denote the matrix made from the left singular vectors of $\widehat{R}$, which correspond to the smallest $(m - \overline{n})$ singular values. Obtain estimates of $\{\lambda_p\}_{p=1}^n$ as the locations of the n dominant peaks of the function

$$h(\lambda) = 1 / \left[f^*(\lambda) \widehat{G} \widehat{G}^T f(\lambda) + f'^*(\lambda) \widehat{G} \widehat{G}^T f'(\lambda) \right]. \quad (22)$$

Let $\lambda = \mu e^{i\omega}$. As $h(\mu e^{i\omega}) = h(\mu e^{-i\omega})$, the maximization of (22) should be done for $\mu \in [0, 1]$ and $\omega \in [0, \pi]$. A direct way of performing this maximization is by a two-dimensional exhaustive search over the previous ranges of μ and ω. A computationally less intensive way is outlined in [6].

5 ESTIMATION OF σ^2

It can be shown, that an estimate $\widehat{\sigma}^2$ of σ^2 can be determined from a simple least-squares technique:

$$\widehat{\sigma}^2 = \left[\sum_{k=1}^{m+\overline{m}-1} \hat{r}(k) \widehat{\varphi}(k) \right] / \left[\sum_{k=1}^{m+\overline{m}-1} \widehat{\varphi}^2(k) \right] \quad (23)$$

where the quantities $\{\widehat{\varphi}(k)\}$ are readily obtained from the estimates of $\{d_k\}$ (we refer to [6] for the details on how to obtain $\{\widehat{\varphi}(k)\}$). The interval for the index k considered in the above equation is motivated by the fact that $\{\hat{r}(k)\}_{k=1}^{m+\overline{m}-1}$ are readily available from the computation of the sample covariance matrix $\widehat{R}$. Whenever $\{\lambda_p\}$ are close to the unit circle, the summation in (23) may be truncated at some value $k < m + \overline{m} - 1$ to slightly enhance the estimation accuracy of $\widehat{\sigma}^2$ (see [6] for details on this aspect).

6 NUMERICAL EXAMPLES

The model used in the simulations is

$$B(z, z^{-1}) = 0.9z^{-2} + 1.33z^{-1} + 2.3 + 1.33z + 0.9z^2 \quad (24)$$

From the factorization (6) we obtain

$$D(z) = 1 + 0.70z^{-1} + 0.90z^{-2}, \ \sigma = 1. \quad (25)$$

The signal-to-noise ratio (SNR) is defined through

$$SNR = 10 \log(\sigma_x^2 / \sigma_w^2) \quad [\text{dB}] \quad (26)$$

where $\sigma_x^2 \triangleq E[x^2(t)]$, with $x(t)$ given by (1).

First we apply the proposed parameter estimation scheme to 20 independent realizations of the above SNARS ($N = 200$, $SNR = 20$dB and $m = \overline{m} = 8$). The results shown in Figure 1 illustrate how the random fluctuations of the location of the estimated zeros of the polynomial $z^2 D(z^{-1})$ affect the shape of the estimated impulse response.

In what follows we discuss the performance of the SNARS-MUSIC methodology in terms of mean and variance of the estimated parameters $\hat{\theta} = (\hat{d}_1, \hat{d}_2, \hat{\sigma})$. In the simulations, these statistics are calculated from 50 independent realizations. We compare the accuracy achieved by SNARS-MUSIC with the Cramer-Rao bound (CRB) corresponding to the ARMA signal model with double poles in (11).

In Figures 2–3 we show the mean and variance of the estimated parameters as a function of the sample size N, for $SNR = 20$dB and $SNR = 25$dB ($m = \overline{m} = 8$). We see that the method proposed here performs well for practical sample sizes and that the estimation of σ is more sensitive to the experimental conditions than the estimation of the coefficients of the D polynomial.

Further numerical results can be found in [6] where we also study the effect of m, $\overline{m}$ and the SNR on the performance of the SNARS-MUSIC parameter estimator.

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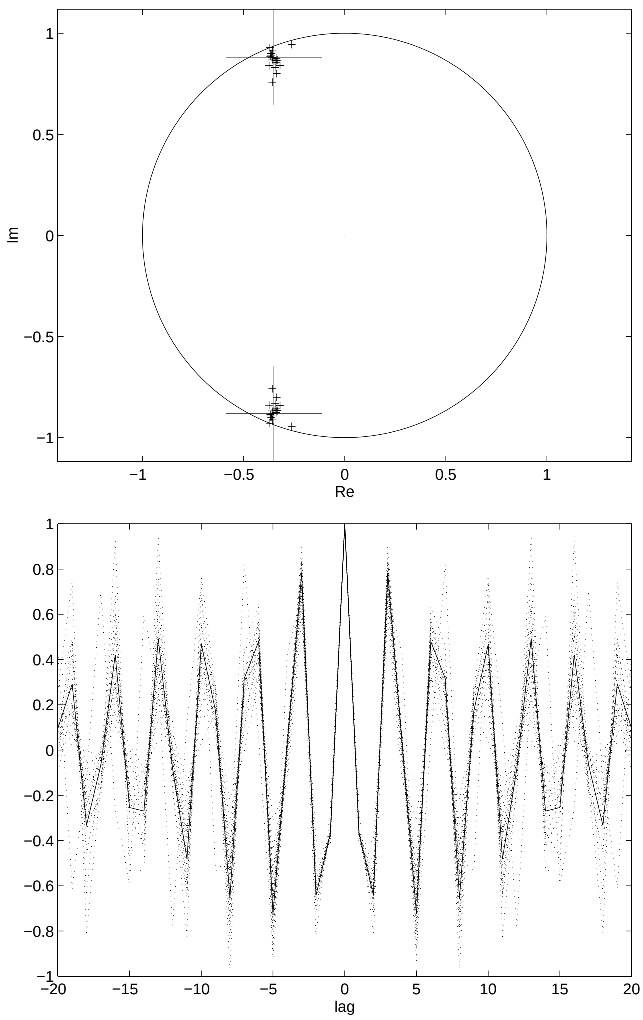

Figure 1: Location of the zeros of $z^2 D(z^{-1})$ and the corresponding (normalized) impulse response, both true (solid) and estimated (dotted), for 20 independent realizations; $SNR = 20$dB, $N = 200$, $m = \overline{m} = 8$.

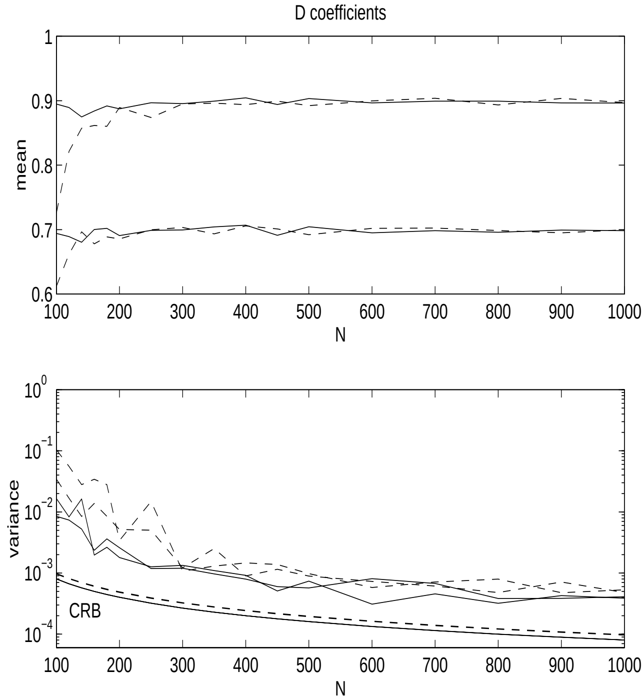

Figure 2: Mean and variance (the CRBs are included for comparison) of the coefficients of D as a function of N for $SNR = 20$ dB (dashed lines) and $SNR = 25$ dB (solid lines), $m = \overline{m} = 8$.

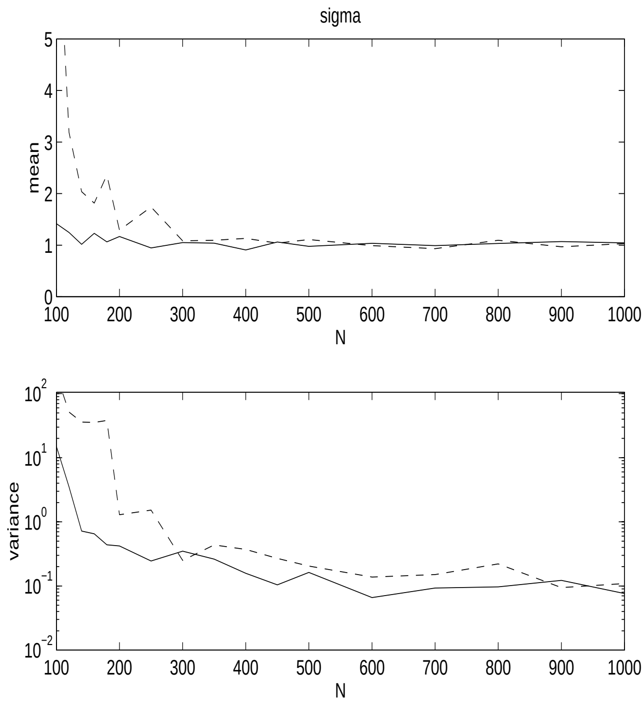

Figure 3: Mean and variance of σ as a function of N for $SNR = 20$ dB (dashed lines) and $SNR = 25$ dB (solid lines), $m = \overline{m} = 8$.