FIR OVERSAMPLED FILTER BANKS AND FRAMES IN $\ell^2(\mathbf{Z})$

Zoran Cvetković and Martin Vetterli

Department of Electrical Engineering and Computer Sciences,
University of California, Berkeley, CA 94720, USA
zoran@eecs.berkeley.edu, martin@eecs.berkeley.edu

ABSTRACT

Perfect reconstruction FIR filter banks implement a particular class of signal expansions in $\ell^2(\mathbf{Z})$. These expansions are studied in this paper. Necessary and sufficient conditions on an FIR filter bank to implement a frame or a tight frame decomposition are given, as well as the necessary and sufficient condition for a feasibility of perfect reconstruction using FIR filters. Complete parameterizations of FIR filter banks satisfying these conditions are given. Further, we study the condition under which the minimal dual frame to the frame associated to an FIR filter bank is also FIR, and give a parameterization of a class of filter banks having this property. We then concentrate on the least constrained class, namely nonsubsampled filter banks, for which these frame conditions have particular forms.

1. INTRODUCTION

The theory of signal expansions into time-frequency localized atoms was a major breakthrough in signal processing in the past decades. One of the important results was the discovery of close relations of the theory of time-frequency expansions in $L^2(\mathbf{R})$ and its analogue in $\ell^2(\mathbf{Z})$, which was being developed independently in the frameworks of filter banks and subband coding. In a number of applications the requirement for orthogonality or linear independence imposes overly restrictive design constraints on expansion vectors. This has motivated development of the theory of time-frequency localized expansions in $L^2(\mathbf{R})$ beyond the orthogonal or biorthogonal case, with a focus on Weyl-Heisenberg and wavelet frames [3]. However, the theory of filter banks has so far been confined mainly to the critically sampled case [1, 2].

This paper studies redundant expansions in $\ell^2(\mathbf{Z})$ which are associated with oversampled FIR filter banks. The issues addressed in this paper are 1) the necessary and sufficient conditions on filter banks for implementing frame or tight frame decompositions in $\ell^2(\mathbf{Z})$ [3], 2) the feasibility of perfect reconstruction using FIR synthesis filters following an FIR analysis, 3) the parameterization of interesting classes of perfect reconstruction FIR oversampled filter banks.

The motivation for the study of oversampled filter banks stems from applications where critical sampling is not needed but rather imposes severe design constraints. Perhaps the most striking example of the limitations of linearly independent expansions comes from the fact that there are no bases for short-time Fourier analysis which are well localized in the time-frequency plane. As soon as some redundancy is introduced the situation changes drastically, so that tight frames for short-time Fourier analysis are attainable [3, 7]. Also, iterated nonsubsampled filter banks allow for a design of highly regular wavelets with a given number of vanishing moments [8]. Nonsubsampled filter banks find their place in applications which require shift invariant signal representations, and also in applications where close approximations of continuous-time transforms are needed.

Notation For a sequence φ, $\tilde{\varphi}$ will denote the sequence which is the complex conjugate of the time-reversed version of φ. When used with matrices of rational functions of the complex variable z, $\tilde{\mathbf{H}}(z)$ will denote the matrix obtained from $\mathbf{H}(z)$ by transposing it, changing all coefficients of the rational functions to their complex conjugates, and substituting z by z^{-1}. If $\tilde{\mathbf{H}}(z) = \mathbf{H}(z)$ we say that $\mathbf{H}(z)$ is parahermitian. A polynomial matrix $\mathbf{H}(z)$ such that $\det\mathbf{H}(z)$ is a nonzero constant is called a unimodular matrix. Note that in this paper we shall use the term *polynomial* for *Laurent polynomials* in general. Complex conjugate transpose of a vector $\mathbf{v}$ will be denoted as $\mathbf{v}^*$.

2. NOTION OF FILTER BANK FRAMES

A family of vectors $\{\varphi_j\}_{j\in J}$ in a Hilbert space $\mathcal{H}$ is said to be a *frame* if and only if there exist positive constants $A > 0,\ B < \infty$ such that

$$A\|f\|^2 \leq \sum_{j\in J} |< f, \varphi_j >|^2 \leq B\|f\|^2$$

for every f in the space. Constants A and B are called *frame bounds*. For a frame $\{\varphi_j\}_{j\in J}$ there exists a frame

$\{\psi_j\}_{j\in J}$ such that every $f \in \mathcal{H}$ can be expanded as

$$f = \sum_{j\in J} < f, \varphi_j > \psi_j \quad \text{or} \quad f = \sum_{j\in J} < f, \psi_j > \varphi_j.$$

If the frame bounds are equal, $A = B$, then $\{\varphi_j\}_{j\in J}$ is said to be a *tight frame* and every $f \in \mathcal{H}$ can be represented as

$$\frac{1}{A}\sum_{j\in J} < f, \varphi_j > \varphi_j.$$

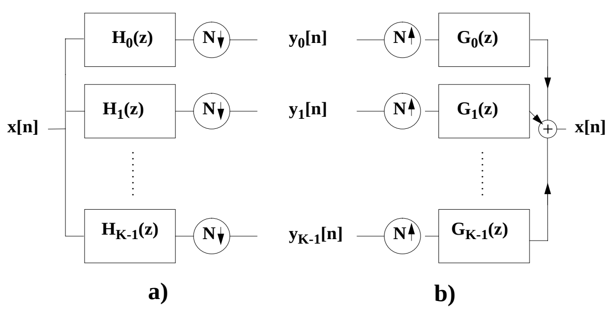

Figure 1: Oversampled filter bank, $K > N$. a) Analysis filter bank. b) Synthesis filter bank.

Consider now a K channel analysis filter bank with subsampling by factor N, $N < K$ (see Figure 1a). With this filter bank we can associate a family of vectors in $\ell^2(\mathbf{Z})$, which has the form

$$\Phi = \{\varphi_{i,j} : \varphi_{i,j}[n] = \varphi_i[n - jN]\}_{i=0,1,\dots,K-1, j\in\mathbf{Z}}. \quad (1)$$

Each of the waveforms φ_i is the time-reversed version of the impulse response of the corresponding filter $H_i(z)$,

$$\varphi_i[n] = \tilde{h}_i[n], \quad i = 0, 1, \dots, K-1. \quad (2)$$

For an input signal $x[n]$ the analysis filter bank calculates its inner products with vectors of the associated family, $y_i[j] =< x, \varphi_{i,j} >$ (see Figure 1a). Every signal in $\ell^2(\mathbf{Z})$ can be perfectly reconstructed in a numerically stable way from these inner products if and only if the family Φ is a frame in $\ell^2(\mathbf{Z})$. If this is satisfied we say that the filter bank implements a *frame decomposition* and we refer to Φ as a *filter bank frame*. A signal $x[n]$ can then be synthesized from the subband components it produces at the output of the filter bank as

$$x[n] = \sum_{i=0}^{K-1} \sum_{j=-\infty}^{\infty} y_i[j]\psi_{i,j}[n], \quad (3)$$

where

$$\Psi = \{\psi_{i,j} : \psi_{i,j}[n] = \psi_i[n - jN]\}_{i=0,1,\dots,K-1, j\in\mathbf{Z}} \quad (4)$$

is another frame in $\ell^2(\mathbf{Z})$. This synthesis formula is implemented by a synthesis filter bank, as the one shown in Figure 1b, if the impulse responses of the filters $G_i(z)$ are equal to the waveforms ψ_i, $g_i[n] = \psi_i[n]$, $\quad i = 0, 1, \dots, K-1$.

3. FRAME CONDITIONS ON FILTER BANKS

Frame conditions on an oversampled filter bank will be expressed in terms of properties of its polyphase analysis matrix. The polyphase analysis matrix, $\mathbf{H_p}(z)$, of a K-channel filter bank with subsampling by factor N is a $K \times N$ matrix $\left[\mathbf{H_p}(z)\right]_{i,j} = H_{i,j}(z)$ where

$$H_{ij}(z) = \sum_{n=-\infty}^{+\infty} h_i[nN - j]z^{-n},$$

is the j-th polyphase component of $H_i(z)$.

We can now establish frame conditions on an FIR filter bank.

Theorem 1 *[6] An FIR filter bank implements a frame decomposition in $\ell^2(\mathbf{Z})$ if and only if its polyphase analysis matrix is of full rank on the unit circle.*

Theorem 2 *[6] An FIR filter bank implements a tight frame decomposition in $\ell^2(\mathbf{Z})$ if and only if its polyphase analysis matrix is paraunitary, $\tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z) = c\mathbf{I}$*

The filter bank implements a frame decomposition if and only if there exists a matrix $\mathbf{G_p}(z)$ of stable rational, not necessarily causal, functions which is a left inverse of its polyphase analysis matrix $\mathbf{H_p}(z)$,

$$\mathbf{G_p}(z)\mathbf{H_p}(z) = c\mathbf{I}. \quad (5)$$

The matrix $\mathbf{G_p}(z)$ satisfying (5) is the so called *polyphase synthesis matrix* and consists of the polyphase components of filters of a synthesis filter bank which can be used for perfect reconstruction from the decomposition obtained from the analysis filter bank. The solution for $\mathbf{G_p}(z)$ of the polyphase equation (5), and hence the synthesis filter bank is not unique in the oversampled case, $K > N$. One solution for $\mathbf{G_p}(z)$ is the pseudoinverse of $\mathbf{H_p}(z)$, given by

$$\mathbf{H_p}^{+}(z) = \left(\tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z)\right)^{-1}\tilde{\mathbf{H}}_\mathbf{p}(z). \quad (6)$$

The frame associated with the filter bank with the polyphase synthesis matrix $\mathbf{G_p}(z) = \mathbf{H_p}^{+}(z)$ is the *minimal dual frame* of the frame associated with the analysis filter bank. For subband signals degraded by additive noise, reconstruction using this filter bank bank reduces to zero the noise component which is orthogonal to the range of the frame expansion. Hence it is important to investigate when both a frame associated with a filter bank and its minimal dual consist of finite length vectors.

Theorem 3 *[6] For a frame associated to an FIR filter bank, with the polyphase analysis matrix $\mathbf{H_p}(z)$, its minimal dual frame consists of finite length vectors if and only if $\tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z)$ is unimodular.*

4. PARAMETERIZATIONS OF FILTER BANK FRAMES

The parameterizations of filter bank frames which are given here are based on the Smith form [4] of corresponding polyphase analysis matrices. Any polynomial matrix $\mathbf{H_p}(z)$ of dimension $K \times N$ $(K > N)$ can be decomposed as the product

$$\mathbf{H_p}(z) = \mathbf{R}(z)\mathbf{D}(z)\mathbf{C}(z), \tag{7}$$

where $\mathbf{R}(z)$ and $\mathbf{C}(z)$ are unimodular matrices of dimensions $K \times K$ and $N \times N$, respectively, while $\mathbf{D}(z)$ is a diagonal $K \times N$ polynomial matrix,

$$\mathbf{D}(z) = \mathrm{diag}\left[d_1(z), ..., d_N(z)\right].$$

The unimodular matrices are products of finitely many elementary matrices, that is

$$\mathbf{R}(z) = \mathbf{R}_1(z)\mathbf{R}_2(z)\cdots\mathbf{R}_m(z)$$

$$\mathbf{C}(z) = \mathbf{C}_1(z)\mathbf{C}_2(z)\cdots\mathbf{C}_n(z).$$

Elementary matrices $\mathbf{R}_i(z)$, $\mathbf{C}_j(z)$ correspond to elementary row (column) operations, and have one of the following forms: 1) permutation matrix, i.e. the identity matrix with two rows permuted, 2) diagonal matrix with elements on the diagonal equal to unity, except for one which is equal to a nonzero scalar, 3) matrix with ones on the main diagonal and a single non-zero entry off the diagonal, which is a polynomial $\alpha(z)$. The unimodular matrices $\mathbf{R}(z)$ and $\mathbf{C}(z)$ can be chosen so that the polynomials $d_i(z)$ are monic and that $d_i(z)$ is a factor of $d_{i+1}(z)$. Such a matrix $\mathbf{D}(z)$ is called the *Smith form* of $\mathbf{H_p}(z)$.

A complete parameterization of FIR filter bank frames follows directly from the Smith form of corresponding polyphase matrices and Theorem 1.

Proposition 1 *[6] An oversampled FIR filter bank implements a frame decomposition in $\ell^2(\mathbf{Z})$ if and only if the polynomials on the diagonal of the Smith form of its polyphase analysis matrix have no zeros on the unit circle.*

A parameterization of FIR filter bank frames having minimal dual frames which are also FIR is given by the following proposition.

Proposition 2 *[6] Consider an oversampled FIR filter bank with the polyphase analysis matrix $\mathbf{H_p}(z)$. $\tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z)$ is unimodular if $\mathbf{H_p}(z)$ has the following form:*

$$\mathbf{H_p}(z) = \mathbf{H}_0\mathbf{R}(z)\mathbf{D}(z)\mathbf{C}(z), \tag{8}$$

where $\mathbf{H}_0$ is a $K \times N$ $(K > N)$ matrix of scalars, such that $\tilde{\mathbf{H}}_0\mathbf{H}_0 = c\mathbf{I}$, $\mathbf{R}(z)$ and $\mathbf{C}(z)$ are products of finitely many elementary matrices, and $\mathbf{D}(z)$ is an $N \times N$ diagonal matrix of polynomials, with nonzero monomials on the diagonal.

Conversely, any unimodular parahermitian matrix of polynomials, $\mathbf{P}(z)$, which is positive definite on the unit circle, can be factored as

$$\mathbf{P}(z) = \tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z),$$

where $\mathbf{H_p}(z)$ is of the form given in (8).

A necessary and sufficient condition for an FIR synthesis is given by the following proposition, which also implicitly gives a complete parameterization of FIR oversampled filter banks in this class.

Proposition 3 *[6] Perfect reconstruction with FIR filters after analysis by an oversampled FIR filter bank is possible if and only if polynomials on the diagonal of the Smith form of the polyphase analysis matrix are monomials.*

Tight filter bank frames are obtained from paraunitary polynomial matrices. A $K \times N$ paraunitary matrix $(K > N)$ can always be embedded into a $K \times K$ [5] paraunitary matrix. The parameterization of the rectangular paraunitary polyphase matrices, that is filter bank tight frames in $\ell^2(\mathbf{Z})$, follows directly from one of the factorizations of square paraunitary matrices studied by Vaidyanathan [1].

Proposition 4 *[6] A $K \times N$ $(K > N)$ polynomial matrix $\mathbf{H_p}(z)$ is paraunitary if and only if it has the decomposition*

$$\mathbf{H_p}(z) = \mathbf{V}_M(z)\mathbf{V}_{M-1}(z)\cdots\mathbf{V}_1(z)\mathbf{H}_0. \tag{9}$$

The elementary building blocks, $\mathbf{V}_i(z)$, have the following form,

$$\mathbf{V}_i(z) = \mathbf{I} - \mathbf{v}_i\mathbf{v}_i^* + z^{-1}\mathbf{v}_i\mathbf{v}_i^*, \tag{10}$$

where $\mathbf{v}_i$ denotes a unit norm vector, while $\mathbf{H}_0$ is a $K \times N$ matrix of scalars such that $\tilde{\mathbf{H}}_0\mathbf{H}_0 = c\mathbf{I}$.

5. NONSUBSAMPLED FILTER BANKS

In the case of nonsubsampled filter banks, the frame conditions as well as the condition for the feasibility of an FIR synthesis after an FIR analysis have a peculiar form. These conditions are reviewed in this section.

The polyphase analysis matrix of a nonsubsampled filter bank (Figure 2a) is a column vector whose entries are the analysis filters themselves,

$$\mathbf{H_p}(z) = [H_0(z)\ H_1(z)\ \ldots\ H_{K-1}(z)]^T. \tag{11}$$

Perfect and stable reconstruction is possible provided that there exist stable filters $G_0(z), G_1(z), \ldots, G_{K-1}(z)$ satisfying

$$H_0(z)G_0(z) + H_1(z)G_1(z) + \ldots + H_{K-1}(z)G_{K-1}(z) = 1. \tag{12}$$

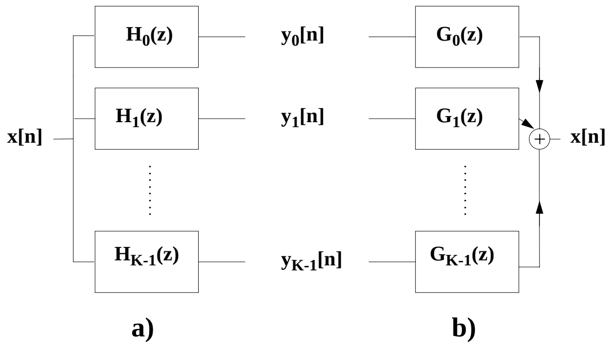

Figure 2: *A K channel nonsubsampled filter bank. a) Analysis filter bank. b) Synthesis filter bank.*

This equation implies reconstruction by a synthesis filter bank with filters $G_i(z)$ (Figure 2b). The necessary and sufficient condition for the existence of such filters is given by the following corollary of Theorem 1.

Corollary 1 *[6] A nonsubsampled filter bank implements a frame decomposition if and only if its analysis filters have no zeros in common on the unit circle.*

The frame condition does not generally guarantee the possibility of FIR reconstruction.

Corollary 2 *[6] Perfect reconstruction using FIR filters after an FIR analysis by a nonsubsampled filter bank is possible if and only if the analysis filters have no zeros in common.*

This result is a corollary of Proposition 3. As a special case of Theorem 2 we have the following result about nonsubsampled filter banks and tight frames.

Corollary 3 *[6] A nonsubsampled filter bank implements a tight frame decomposition if and only if its analysis filters are power complementary:*

$$H_0(z)\tilde{H}_0(z)+H_1(z)\tilde{H}_1(z)+...+H_{K-1}(z)\tilde{H}_{K-1}(z) = 1. \quad (13)$$

In the case of nonsubsampled filter banks, a frame associated with an FIR filter bank has a minimal dual frame consisting of finite length vectors only if the frame is tight. This result is an immediate corollary of Theorem 3.

Corollary 4 *[6] For a frame associated with an FIR nonsubsampled filter bank, its minimal dual frame consists of finite length vectors if and only if the analysis filters are power complementary, that is if and only if the frame is tight.*

6. CONCLUSION

In this paper properties of oversampled FIR filter banks are studied. Necessary and sufficient conditions on a filter bank to implement a frame or a tight frame decomposition in $\ell^2(\mathbf{Z})$ were given in terms of properties of the corresponding polyphase analysis matrix. Complete parameterizations of filter bank frames and tight frames were also given. A necessary and sufficient condition for the feasibility of perfect reconstruction with FIR filters was also established, as well as a necessary and sufficient condition for an FIR filter bank frame to have an FIR dual. Filter banks in these two classes were also parameterized. In the case of nonsubsampled filter banks these conditions have particular forms, which were also discussed.

7. REFERENCES

[1] P. P. Vaidyanathan, "Multirate Systems and Filter Banks", Prentice-Hall, Englewood Cliffs, New Jersey, 1993.

[2] M. Vetterli and J. Kovačević, "Wavelets and Subband Coding", Prentice-Hall, Englewood Cliffs, New Jersey, 1995

[3] I. Daubechies,"The Wavelet Transform, Time-Frequency Localization and Signal Analysis", *IEEE Trans. on Information Theory*, Vol.36, No.5, September 1990, pp.961-1005.

[4] H. J. S. Smith, On Systems of Linear Indeterminate Equations and Congruences, *Philos. Trans. Royal Soc. London*, Vol.151, 1861, pp.293-326.

[5] H. Park, "A Computational Theory of Laurent Polynomial Rings and Multidimensional FIR Systems", Ph. D. Dissertation, University of California at Berkeley, May 1995.

[6] Z. Cvetković and M. Vetterli, "Oversampled Filter Banks," *to appear in IEEE Trans. Signal Processing.*

[7] Z. Cvetković,"Oversampled Modulated Filter Banks and Tight Gabor Frames in $\ell^2(\mathbf{Z})$", *Proc. ICASSP '95*, vol. 2, May 1995, pp. 1456-1459.

[8] Z. Cvetković and M. Vetterli,"Discrete-Time Wavelet Extrema Representation: Design and Consistent Reconstruction", *IEEE Trans. Signal Processing*, vol. 43, No. 3, March 1995, pp. 681-693.