SVD-ICA: A NEW TOOL TO ENHANCE THE SEPARATION BETWEEN SIGNAL AND NOISE SUBSPACES

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ABSTRACT

In multisensor signal processing (geophysics, underwater acoustic, etc.), the Singular Value Decomposition (SVD) is a useful tool to perform a separation of the initial dataset into two complementary subspaces. The SVD of the data matrix $\{x, t\}$ provides two orthogonal matrices that convey information on propagation vectors and normalized wavelets. The constraint imposed by the orthogonality's condition for the propagation vectors introduce errors in the signal subspace. To relax this condition, another matrix of normalized wavelet is calculated exploiting the concept of Independent Component Analysis (ICA). Efficiency of this new separation tool using the combined SVD-ICA procedure is shown on realistic dataset.

1 Introduction

The main goal of SVD in multisensor signal processing is to obtain a decomposition of the initial dataset into two complementary subspaces called signal and noise subspace [5]. In some applications [8], the SVD is used to separate waves with a high and low degree of sensor-to-sensor correlation. In geophysics for example, in Vertical Seismic Profiling in particular [4], this tool is used to define a low-pass, a band-pass and a high-pass SVD eigenimages, in terms of the range of singular values and therefore to make a separation between the downgoing wavefield, the upgoing wavefield and the noise.

For signals recorded on multisensors, the SVD of the space-time data matrix $\{x, t\}$ provides two orthogonal matrices that convey information on propagation vectors and normalized wavelets, and one pseudodiagonal matrix of singular values. Since the propagation vectors are, generally, not orthogonal, we propose to associate the SVD procedure with the Independent Component Analysis (ICA). In fact, we construct a new basis where the normalized waves are independent at the fourth-order and, in the same time, we relax the orthogonality's condition for the propagation vectors. Application of this new separation tool on data is shown in the last section, where we illustrate the efficiency of the combined SVD-ICA procedure.

2 Singular Value Decomposition

The SVD is a powerful decomposition in matrix computation. The basic approach for computing the SVD is given in [6].

Let $s_i(t)$ be the signal received on the i^{th} sensor. With signals sampled in time $s_{ij} = s_i(jT_e)$, where $i = 1..N_x$, N_x the number of sensors and T_e the time sample ratio, we can write the received signals in a data matrix (in generally, the number of time samples N_t is greater than N_x):

$$S = \{s_{ij}\} \in R^{N_x \times N_t} \quad (1)$$

The SVD of data matrix S is given by [6]:

$$S = U\Lambda V^T = \sum_{k=1}^{N} \lambda_k u_k v_k^T \quad (2)$$

where $U = [u_1, .., u_k, .., u_{N_x}]$ is an $N_x \times N_x$ orthonormal matrix made up of left singular vectors $u_k \in R^{N_x}$, $V = [v_1, .., v_k, .., v_{N_t}]$ is an $N_t \times N_t$ orthonormal matrix made up of right singular vectors $v_k \in R^{N_t}$, and Λ is a $N_x \times N_t$ pseudodiagonal matrix, $\Lambda = diag\,(\lambda_1, .., \lambda_k, .., \lambda_N)$ made up of singular values $\lambda_k \in R^+$, with the diagonal entries ordered $\lambda_1 \geq ... \geq \lambda_N \geq 0$. The superscript T indicates transpose and $N = \min(N_x, N_t)$.

The product $u_k v_k^T$ is an $N_x \times N_t$ unitary rank matrix named the k^{th} eigenimage of data matrix S [1]. Therefore, the received data S is given by the sum of eigenimages, pondered by the correspondent singular value. Let note $u_k = [u_{k1}, .., u_{ki}, .., u_{kN_x}]^T$ and $v_k = [v_{k1}, .., v_{kj}, .., v_{kN_t}]^T$ the k^{th} left, respectively right, singular vector. Sample data s_{ij} at time j on sensor i is expressed as:

$$s_{ij} = \sum_{k=1}^{N} \lambda_k u_{ki} v_{kj} \quad (3)$$

- v_{ki} gives the time dependance, hence the right singular vector v_k is named normalized wavelets,
- u_{ki} gives the amplitude in real case (amplitude and phase in complex case), thus the left singular vector u_k is named propagation vector.

In the noise free case, if the recorded signals are linearly dependent, for example if they are equal to within a scale factor, the matrix S is of rank one and the perfect reconstruction requires only the first eigenimage [4]. If the N_x recorded signals are linearly independent, the matrix S is full rank and the perfect reconstruction requires all eigenimages.
Hence, in practice, before initiating the SVD calculation, a time correction operation on the received data is applied to obtain a waves alignment (infinite apparent velocity of propagation on the array sensor), which allows a decomposition with a smaller space [7].

3 Signal and noise subspaces

Supposing the noise independent from one sensor to other, the separation between the signal and noise subspace, after the time correction operation, is given by:

$$S = S_s^P + S_n^P = \sum_{k=1}^{P} \lambda_k u_k v_k^T + \sum_{k=P+1}^{N} \lambda_k u_k v_k^T \qquad (4)$$

where the first P eigenimages of data matrix represent the signal subspace S_s^P and the other $N-P$ eigenimages represent the noise subspace S_n^P. The choice of P depends on the relative magnitudes of the singular values. For example, if we have only a non-dispersive source wave with infinite apparent velocity of propagation on the array sensor, the rank P that describe the signal subspace will be equal to 1.
Furthermore, we can make a separation of the signal subspace into two orthogonal subspaces named low-pass S_s^{LP}, respectively band-pass S_s^{BP} eigenimages [4]:

$$S_s^P = S_s^{LP} + S_s^{BP} = \sum_{k=1}^{Q} \lambda_k u_k v_k^T + \sum_{k=Q+1}^{P} \lambda_k u_k v_k^T \qquad (5)$$

where the low-pass eigenimage, associated to the higher singular values, contains the source waves with a high degree of sensor-to-sensor correlation, the band-pass eigenimage, associated to the lower singular values in the signal subspace, is constructed by events with a weak degree of sensor-to-sensor correlation, and Q is the number of the highly correlated source waves depending on the relative magnitudes of the singular values. In geophysics for example, in Vertical Seismic Profiling, the low-pass eigenimage usually contains the downgoing waves and the band-pass eigenimage contains the upgoing waves.

4 SVD and ICA

The normalized wavelets that describe the signal subspace $[v_1, .., v_k, .., v_P] \overset{not}{=} V_P$ are orthogonal, therefore statistically independent at the second-order. Also, by construction, the propagation vectors that describe the signal subspace $[u_1, .., u_k, .., u_P] \overset{not}{=} U_P$ are orthogonal. There is no physical reason for which the propagation vectors u_k are orthogonal. Furthermore, we can have waves in the dataset for which the propagation vectors are not orthogonal. In this case, imposing the criterion of orthogonality for the left matrix, we'll force the normalized wavelets in the right matrix to be a mixture of source waves.
To relax this constrain, the idea is to find another matrix of normalized wavelets $[\widetilde{v}_1, .., \widetilde{v}_k, .., \widetilde{v}_P] \overset{not}{=} \widetilde{V}_P$ for which these waves are "the most independent possible". This can be made with the Independent Component Analysis (ICA).
ICA can be solved by a two-stage algorithm, consisting of a prewithening and a high-order step [9]. The first step is accomplished by the SVD since the normalized wavelets in V_P are statistically independent at the second-order. The second step consist in finding a $P \times P$ rotation [unitary] matrix B for which the components of $\widetilde{V}_P = V_P B$ are independent. Denoting by Γ the family of cumulants of V_P and by K those of $\widetilde{V}_P$, the rotation matrix B is given by approximate diagonalization of the sample cumulant [3]:

$$K_{ijkl} = \sum_{pqrs} B_{ip} B_{jq} B_{kr} B_{ls} \Gamma_{pqrs} \qquad (6)$$

The way to solve this system is, for example, Joint Approximate Diagonalization of Eigenmatrices (JADE) [2] or Maximal Diagonality (MD) [3]. With $V_P{}^T = B\widetilde{V}_P^T$, the signal subspace given in (4) can be rewritten:

$$S_s^P = U_P \Lambda_P V_P^T = U_P \Lambda_P B \widetilde{V}_P^T = C_P \widetilde{V}_P^T \qquad (7)$$

From the matrix $C_P = [c_1, .., c_k, .., c_P]$, we can obtain two matrices. One is a $N_x \times P$ matrix $\widetilde{U}_P = [\widetilde{u}_1, .., \widetilde{u}_k, .., \widetilde{u}_P]$ with normalized columns $\widetilde{u}_k = c_k / \|c_k\|$ and the second is a $P \times P$ diagonal matrix $\Delta_P = diag\,(\delta_1, .., \delta_P)$ with the diagonal inputs $\delta_k = \|c_k\|$.
The inputs in the diagonal matrix are generally not ordered. Hence, we perform a permutation between the columns of $\widetilde{U}_P$ as well as the columns of $\widetilde{V}_P$ to order the elements of Δ_P. The signal subspace is given by:

$$S_s^P = \widetilde{U}_P \Delta_P \widetilde{V}_P^T = \sum_{k=1}^{P} \delta_k \widetilde{u}_k \widetilde{v}_k^T \qquad (8)$$

Therefore, in this decomposition we have relaxed the condition of orthogonality for the propagation vectors $\widetilde{u}_k$ and, in the same time, imposed a stronger criterion for the normalized wavelets $\widetilde{v}_k$, i.e. to be statistically independents at the fourth-order. Supposing the diagonal inputs ordered, i.e. $\delta_1 \geq ... \geq \delta_P \geq 0$, the low-pass $\widetilde{S}_s^{LP}$ and the band-pass $\widetilde{S}_s^{BP}$ eigenimages using SVD-ICA are:

$$S_s^P = \widetilde{S}_s^{LP} + \widetilde{S}_s^{BP} = \sum_{k=1}^{Q_1} \delta_k \widetilde{u}_k \widetilde{v}_k^T + \sum_{k=Q_1+1}^{P} \delta_k \widetilde{u}_k \widetilde{v}_k^T \quad (9)$$

where Q_1 is the number of the highly correlated source waves depending on the relative magnitudes of the elements in the diagonal matrix Δ_P.

5 Application on synthetic data

The recorded signals S on array sensor ($N_x = 8$), after the time correction operation, are represented in figure 1. On this figure we have 2 source waves: one wave

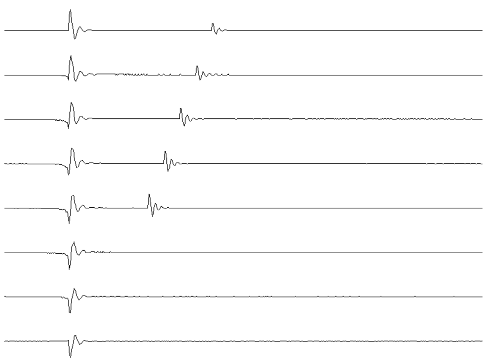

Figure 1: The received signals S

with infinite apparent velocity of propagation on the array (arriving at the same time on each sensor) and one wave with a negative apparent velocity of propagation on the array, which is present between sensor 1 and 5. On each sensor we have applied an amplitude reduction and a phase rotation to simulate the absorption phenomena and the dispersion effect. The samples number is $N_t = 512$ and the mean signal-to-noise ratio is $SNR_{mean} = 30dB$.

The singular values given by SVD are presented in figure 2. The number of singular values used to describe

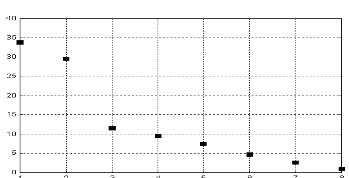

Figure 2: The relative magnitudes in Λ

the signal subspace is $P = 7$, among the first two are related to the highest correlated waves ($Q = 2$).

We can see in figure 3 that the first two normalized wavelets in V_P are a mixture of the source waves.

The low-pass eigenimage S^{LP} is presented in figure

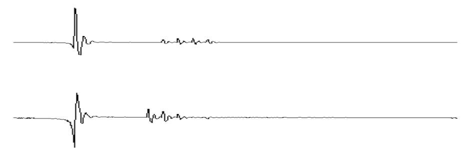

Figure 3: The first 2 normalized wavelets in V_P

4 and the band-pass eigenimage S^{BP} in figure 5. It is clear from these figures that the classical SVD implies some errors in the low-pass and band-pass eigenimages for a wavefield separation objective.

In figure 6 are presented the relative magnitudes of

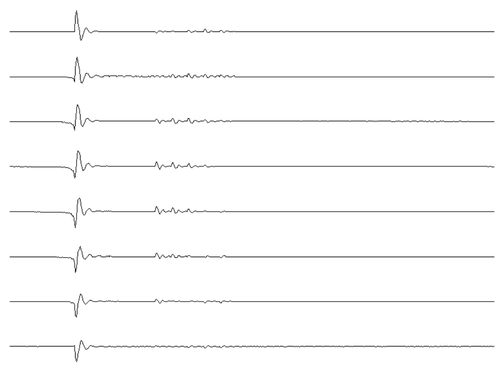

Figure 4: The low-pass eigenimage S^{LP}

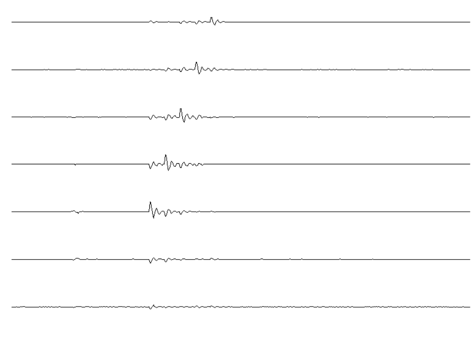

Figure 5: The band-pass eigenimage S^{BP}

the elements in the diagonal matrix Δ_P obtained using the combined SVD-ICA on the signal subspace ($P = 7$) and also the last singular value given by SVD, related to the noise subspace. The number of components that describes the high correlated waves is $Q_1 = 2$.

The first two normalized wavelets in $\widetilde{V}_P$ presented in figure 7 are clearly nearest to the original waves than the normalized wavelets given by SVD.

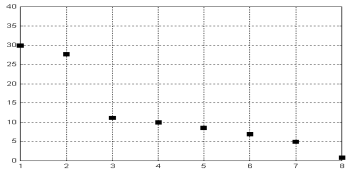

Figure 6: The relative magnitudes in Δ_P

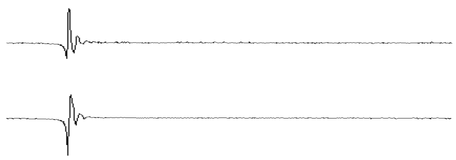

Figure 7: The first 2 normalized wavelets in $\widetilde{V}_P$

The low-pass eigenimage $\widetilde{S}^{LP}$ is given in figure 8 and the band-pass eigenimage $\widetilde{S}^{BP}$ in figure 9. The low-pass eigenimage extracts the first highly correlated wave without any visible interference with the other waves. The residual errors presented in classical SVD are eliminated. This improvement is due to the fact that by ICA we have imposed a fourth-order independence condition stronger than decorrelation used in classical SVD.

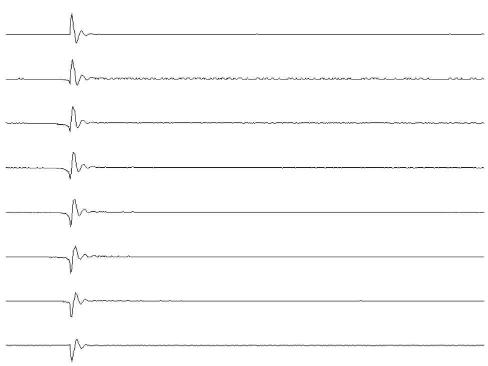

Figure 8: The low-pass eigenimage $\widetilde{S}^{LP}$

6 Conclusions

We have presented a new method for the separation of the multidimensional signal space. The classical SVD imposes the orthogonality for the propagation vectors and forces the normalized wavelets in the left singular matrix to be a mixture of source waves. Using the combined SVD-ICA, the residual errors presented in classical SVD are eliminated. This improvement is due to the fact that by ICA we have imposed a fourth-order independence condition stronger than decorrelation used in classical SVD. With this decomposition method we

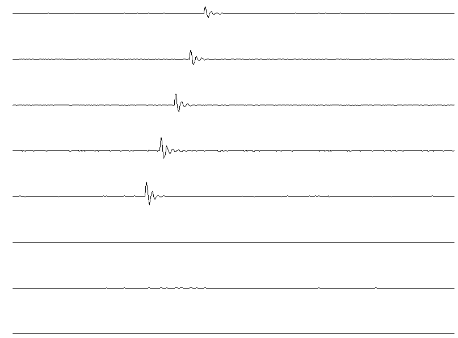

Figure 9: The band-pass eigenimage $\widetilde{S}^{BP}$

relax the non physically justified orthogonality of the propagation vectors u_k.

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