ASYMPTOTICALLY MINIMUM VARIANCE SECOND-ORDER ESTIMATION OF FREQUENCIES FOR MIXED SPECTRA TIME SERIES

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ABSTRACT

This paper addresses asymptotically minimum variance (AMV) algorithms in the class of algorithms based on second-order statistics for estimating frequencies of mixed complex circular processes. To reduce the computational complexity due to the nonlinear minimization required by the matching approach, the covariance matching estimation techniques (COMET) is included in the algorithm. Numerical examples illustrate the performance of the AMV algorithm.

1 Introduction

There is considerable literature about second-order algorithms for recovering sinusoids embedded in white noise. However, few contributions have been devoted to the case of correlated noise, such as the modified Pisarenko decomposition (MPD) of [1].

To improve the performance of these algorithms, we propose in this paper to consider asymptotically (in the number of measurements) minimum variance algorithms in the class of algorithms based on second-order moments. Porat and Friedlander [2] were the first to derive such estimators for estimating the MA and ARMA parameters of non-Gaussian processes from sample high-order statistics. Then, this approach was used for blind channel estimation (see e.g., the works by Giannakis and Halford [3]) and for DOA estimation (see e.g., the works by Ottersten *et al.* [4]) using second-order statistics.

The paper is organized as follows. Section 2 presents the asymptotically minimum variance second-order estimator for stationary complex circular processes with a special attention to the statistics involved. As an application, the estimation of the frequencies of sinusoids for mixed spectra time series containing a sum of sinusoids and an MA process is considered in Section 3. The asymptotic performance and the robustness to error model are analyzed in Section 4. Finally, illustrative examples with comparisons with the MPD estimator are given in Section 5.

2 Asymptotic minimum variance second-order estimator

We consider a strict-sense stationary complex circular process x_t whose $M \times M$ Hermitian Toeplitz structured covariance matrix $\mathbf{R}(\Theta) = \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^H)$ with $\mathbf{x}_t \stackrel{\text{def}}{=} (x_t, \ldots, x_{t-M+1})^T$, is parameterized by the real parameter $\Theta \in \mathcal{R}^L$. This parameter is supposed identifiable from $\mathbf{R}(\Theta)$.

The covariance matrix $\mathbf{R}(\Theta)$ is classically estimated by $\mathbf{R}_T = \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t^H$ or by the Hermitian Toeplitz matrix $\mathbf{R}_T^{\text{to}}$ built from its first column $\mathbf{r}_T = \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t x_t^*$.

To extend the ideas of Porat and Friedlander [2] concerning asymptotically minimum variance second-order estimators, to complex circular processes, two conditions must be satisfied. First, the covariance $\mathbf{C}_R(\Theta)$ of the asymptotic distribution of $\mathbf{R}_T$ must be regular. Second, the involved second-order algorithm considered as a mapping which associates to $\mathbf{R}_T$, the estimate Θ_T $\mathbf{R}_T \to \Theta_T = \mathrm{alg}(\mathbf{R}_T)$ must be differentiable w.r.t. $\mathbf{R}_T$ at the point $\mathbf{R}(\Theta)$. While, the second condition is satisfied, the first one is not. In fact, it is proved in [5], that the asymptotic distributions of $\mathbf{R}_T$ and $\mathbf{R}_T^{\text{to}}$ have the same covariance matrix $\mathbf{C}_R(\Theta)$. Therefore, the rank of $\mathbf{C}_R(\Theta)$ is the rank of the set of the entries of $\mathbf{R}_T^{\text{to}}$ considered as random variables, i.e. $\mathrm{rank}(\mathbf{C}_R(\Theta)) = 2M-1$. To solve this difficulty, we could work only with the first column $\mathbf{r}_T$ of $\mathbf{R}_T$ because $\mathbf{r}(\Theta)$ contains the same information about Θ as $\mathbf{R}(\Theta)$. But this choice leads to an algorithm $\mathbf{r}_T \to \Theta_T = \mathrm{alg}(\mathbf{r}_T)$ that is not differentiable w.r.t. $\mathbf{r}_T$ at point $\mathbf{r}(\Theta)$.

To make this algorithm differentiable, we consider in the following the statistics $\mathbf{s}_T$ constituted by $\mathbf{r}_T$ and $\mathbf{r}_T^*$ where the first common real term $r_{0,T}$ appears only once, i.e. $\mathbf{s}_T \stackrel{\text{def}}{=} \begin{pmatrix} r_{0,T} \\ \mathbf{r}'_T \\ \mathbf{r}'^*_T \end{pmatrix}$ with $\mathbf{r}_T \stackrel{\text{def}}{=} \begin{pmatrix} r_{0,T} \\ \mathbf{r}'_T \end{pmatrix}$,

So for $\delta\mathbf{s}$ structured as $\delta\mathbf{s} = \begin{bmatrix} \delta r_{0,T} \\ \delta\mathbf{r}'_T \\ \delta\mathbf{r}'^*_T \end{bmatrix}$, if $\mathbf{D}_{r_0}^{\text{alg}}$ and $\mathbf{D}_{\mathbf{r}'}^{\text{alg}}$ are respectively defined by $\frac{\partial \text{alg}}{\partial r_0}$ and $[\mathbf{D}_{\mathbf{r}'}^{\text{alg}}]_{k,l} \stackrel{\text{def}}{=}$

$\frac{1}{2}\left[\frac{\partial \text{alg}_k}{\partial \Re(\mathbf{r}'(\Theta))_l} - i\frac{\partial \text{alg}_k}{\partial \Im(\mathbf{r}'(\Theta))_l}\right]$, it is straightforward to see that:

$$\begin{aligned}\text{alg}[\mathbf{s}(\Theta)+\delta\mathbf{s}] &= \text{alg}[\mathbf{s}(\Theta)] + [\mathbf{D}^{\text{alg}}_{r_0}, \mathbf{D}^{\text{alg}}_{\mathbf{r}'}, \mathbf{D}^{\text{alg}}_{\mathbf{r}'^*}]\delta\mathbf{s} + o(\delta\mathbf{s}) \\ &= \Theta + \mathbf{D}^{\text{alg}}_{\mathbf{s}}\delta\mathbf{s} + o(\delta\mathbf{s})\end{aligned}$$

with $\mathbf{D}^{\text{alg}}_{\mathbf{s}} \stackrel{\text{def}}{=} [\mathbf{D}^{\text{alg}}_{r_0}, \mathbf{D}^{\text{alg}}_{\mathbf{r}'}, \mathbf{D}^{\text{alg}}_{\mathbf{r}'^*}]$. Therefore $\mathbf{D}^{\text{alg}}_{\mathbf{s}}$ is a left inverse of $\mathbf{S} \stackrel{\text{def}}{=} \frac{\partial \mathbf{s}(\Theta)}{\partial \Theta}$:

$$\mathbf{D}^{\text{alg}}_{\mathbf{s}}\mathbf{S} = \mathbf{I}_L, \qquad (1)$$

and this time, the rank of the set of the entries of $\mathbf{s}_T$ is $2M-1$ and so, the covariance $\mathbf{C}_s(\Theta)$ of the asymptotic distribution of $\mathbf{s}_T$ is a Hermitian positive definite matrix. Therefore, by application of theorem 2 of [2], extended to the complex case, we obtain:

Theorem 1 *The asymptotic covariance of an estimator of Θ given by an arbitrary second-order algorithm is bounded below by $(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}$:*

$$\mathbf{C}_\Theta = \mathbf{D}^{\text{alg}}_{\mathbf{s}}\mathbf{C}_s(\Theta)(\mathbf{D}^{\text{alg}}_{\mathbf{s}})^H \geq (\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}. \qquad (2)$$

Furthermore, thanks to a straightforward extension of theorem 3 of [2], this lowest bound is asymptotically tight, i.e., there exists an algorithm alg(.) whose covariance of the asymptotic distribution of Θ_T satisfies (2) with equality. This algorithm is given by the following nonlinear least mean square algorithm:

$$\Theta_T = \arg\min_\alpha [\mathbf{s}_T - \mathbf{s}(\alpha)]^H\mathbf{C}_s^{-1}(\Theta)[\mathbf{s}_T - \mathbf{s}(\alpha)]. \qquad (3)$$

In practice, it is difficult to optimize this nonlinear function where it involves the computation of $\mathbf{C}_s^{-1}(\Theta)$. Porat and Friedlander proved in [6] that the lowest bound (2) is also obtained if an arbitrary consistent estimate $\mathbf{C}_{s,T}$ of $\mathbf{C}_s(\Theta)$ is used in (3). So this minimization can be preferably replaced by the following

$$\Theta_T = \arg\min_\alpha [\mathbf{s}_T - \mathbf{s}(\alpha)]^H\mathbf{C}_{s,T}^{-1}[\mathbf{s}_T - \mathbf{s}(\alpha)]. \qquad (4)$$

3 Application to the estimation of frequencies for mixed spectra time series

In the following, we will be concerned with the complex valued process $x_t = s_t + n_t$, with

$$s_t \stackrel{\text{def}}{=} \sum_{k=1}^K a_k e^{i2\pi f_k t} e^{i\phi_k} \quad \text{and} \quad n_t \stackrel{\text{def}}{=} \sum_{q=0}^Q b_q u_{t-q}.$$

$(u_t)_{t=1,...}$ is a sequence of circular complex zero-mean i.i.d. random variables where $\text{E}|u_t^4| < \infty$, with $\kappa_u \stackrel{\text{def}}{=} \text{Cum}(u_t, u_t^*, u_t, u_t^*)$. $(a_k)_{k=1,...,K}$ and $(b_q)_{q=0,...,Q}$ are unknown fixed real and complex numbers respectively. $(f_k)_{k=1,...,K}$ are unknown fixed distinct real numbers in $]-1/2, +1/2[$, ϕ_k are random variables uniformly distributed on $[0, 2\pi]$ and $(\phi_k)_{k=1,...,K}$ and u_t are mutually independent. In this case, x_t is a complex circular strict-sense stationary process. Its $M \times M$ covariance matrix is given by:

$$\mathbf{R}(\Theta) = \sum_{k=1}^K a_k^2 \mathbf{e}(f_k)\mathbf{e}^H(f_k)$$

$$+ \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_Q & 0 & \cdots & 0 \\ \gamma_1^* & \gamma_0 & \cdots & \gamma_{Q-1} & \gamma_Q & \ddots & \cdots \\ \vdots & \ddots & \ddots & \gamma_{Q-2} & \gamma_{Q-1} & \ddots & 0 \\ \gamma_Q^* & \gamma_{Q-1}^* & \ddots & \ddots & \ddots & \ddots & \gamma_Q \\ 0 & \gamma_Q^* & \gamma_{Q-1}^* & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \gamma_1^* & \gamma_0 & \gamma_1 \\ 0 & \cdots & 0 & \gamma_Q^* & \cdots & \gamma_1^* & \gamma_0 \end{pmatrix}$$

with $\mathbf{e}(f_k) \stackrel{\text{def}}{=} (1, e^{i2\pi f_k}, \dots, e^{i2(M-1)\pi f_k})^H$ and $(\gamma_k)_{k=0,...,Q} \stackrel{\text{def}}{=} \text{E}(n_t n_{t-k}^*)$.

$\mathbf{R}(\Theta)$ is parametrized by the $L = 2(Q+K)+1$ real parameters $\Theta = (\Theta_1, \Theta_2)$ with $\Theta_1 \stackrel{\text{def}}{=} (f_1, \dots, f_K)^T$ and $\Theta_2 \stackrel{\text{def}}{=} [a_1^2, .., a_K^2, \gamma_0, \Re(\gamma_1), .., \Re(\gamma_Q), \Im(\gamma_1), .., \Im(\gamma_Q)]^T$. We note that the first column $\mathbf{r}(\Theta)$ of $\mathbf{R}(\Theta)$ is linear with respect to Θ_2: $\mathbf{r}(\Theta) = \mathbf{\Psi}'(\Theta_1)\Theta_2$ with

$$\mathbf{\Psi}'(\Theta_1) = \begin{pmatrix} \mathbf{e}(f_1), \dots, \mathbf{e}(f_K), & \begin{matrix} \mathbf{I}_{Q+1}, & \begin{matrix} \mathbf{0}^T \\ -i\mathbf{I}_Q \end{matrix} \\ \multicolumn{2}{c}{\mathbf{O}_{M-Q-1,2Q+1}} \end{matrix} \end{pmatrix}.$$

Therefore $\mathbf{\Psi}'(\Theta_1)$ has column full rank (over the field $\mathcal{R}$) iif $M-Q-1 \geq K$. This condition is equivalent to having the number of unknown real parameters no larger than the number of estimating equations available, i.e., $2(Q+K)+1 \leq 1+2(M-1)$. This necessary condition is not sufficient to ensure identifiability. We suppose in this paper, that this condition is satisfied and that it implies the identifiability at the true value of Θ. The equivalent statistics $\mathbf{s}(\Theta)$ can be written as:

$$\mathbf{s}(\Theta) = \mathbf{\Psi}(\Theta_1)\Theta_2 \qquad (5)$$

where $\mathbf{\Psi}(\Theta_1)$ is also a full column rank matrix (over the field $\mathcal{R}$). The minimization (4) with respect to Θ_2 is immediate if Θ_2 is not restricted to be real. With a geometric procecedure, we obtain:

$$\widehat{\Theta}_2 = [\mathbf{\Psi}^H(\Theta_1)\mathbf{W}\mathbf{\Psi}(\Theta_1)]^{-1}\mathbf{\Psi}^H(\Theta_1)\mathbf{W}\mathbf{s}_T \qquad (6)$$

with $\mathbf{W} \stackrel{\text{def}}{=} \mathbf{C}_{s,T}^{-1}$. With arguments similar to that of COMET [4], we prove that $\widehat{\Theta}_2$ is real-valued.

Proof If $\mathbf{J} \stackrel{\text{def}}{=} \frac{1}{2}\begin{pmatrix} 2 & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{I}_{M-1} & \mathbf{I}_{M-1} \\ \mathbf{0} & -i\mathbf{I}_{M-1} & i\mathbf{I}_{M-1} \end{pmatrix}$ denotes the linear inversible transformation that associates to $\mathbf{s}_T$, the real-valued vector constituted by the real and imaginary parts of $\mathbf{s}_T$, $\widehat{\Theta}_2$ given by (6) assumes the form: $[(\mathbf{J\Psi})^H(\mathbf{JW}^{-1}\mathbf{J}^H)^{-1}(\mathbf{J\Psi})]^{-1}(\mathbf{J\Psi})^H(\mathbf{JW}^{-1}\mathbf{J}^H)^{-1}\mathbf{J}\mathbf{s}_T$, where $\mathbf{J}\mathbf{s}_T$ is real and so is $\mathbf{J\Psi}$. It remains to examine $\mathbf{JW}^{-1}\mathbf{J}^H$. With $\mathbf{W}^{-1}$ structured as in (9), $\mathbf{JW}^{-1}\mathbf{J}^H$ assumes the form

$$\frac{1}{2}\begin{pmatrix} 2c & 2\Re(\mathbf{c}^T) & 2\Im(\mathbf{c}^T) \\ 2\Re(\mathbf{c}) & \Re(\mathbf{C}_{r'}) + \Re(\mathbf{C}'_{r'}) & \Im(\mathbf{C}_{r'}^T) + \Im(\mathbf{C}'^T_{r'}) \\ 2\Im(\mathbf{c}) & \Im(\mathbf{C}_{r'}) + \Im(\mathbf{C}'_{r'}) & \Re(\mathbf{C}_{r'}) - \Re(\mathbf{C}'_{r'}) \end{pmatrix}$$

which is real-valued. ∎

So that $\widehat{\Theta}_2$ given by (6) is the real value that minimizes (4). $\Theta_{1,T}$ is obtained by subtituting $\widehat{\Theta}_2$ in (4):

$$\Theta_{1,T} = \arg\max_{\alpha} V(\alpha) \tag{7}$$

with
$V(\alpha) \stackrel{\text{def}}{=} \mathbf{s}_T^H \mathbf{W}\boldsymbol{\Psi}(\alpha)[\boldsymbol{\Psi}^H(\alpha)\mathbf{W}\boldsymbol{\Psi}(\alpha)]^{-1}\boldsymbol{\Psi}^H(\alpha)\mathbf{W}\mathbf{s}_T.$

4 Performance analysis

4.1 Asymptotic minimum variance

The asymptotic minimum variance of the parameters is based on the following theorem that is proved in [5]

Theorem 2 *The first columns $\mathbf{r}_T$ and $\mathbf{r}_T^{to}$ of respectively $\mathbf{R}_T$ and $\mathbf{R}_T^{to}$ converge in distribution to the same zero-mean complex Gaussian distribution of covariances $(\mathbf{C}_r, \mathbf{C}'_r)$.*

$$\sqrt{T}\ \ (\mathbf{r}_T - \mathbf{r}(\Theta)) \xrightarrow{\mathcal{L}} \mathcal{N}_c(\mathbf{0}; \mathbf{C}_r, \mathbf{C}'_r). \tag{8}$$

where

$$\begin{aligned}
\mathbf{C}_r &= \int_{-1/2}^{+1/2} S_n^2(f)\mathbf{e}(f)\mathbf{e}^H(f)df \\
&+ 2\sum_{k=1}^{K} a_k^2 S_n(f_k)\mathbf{e}(f_k)\mathbf{e}^H(f_k) + \kappa_u \gamma\gamma^H, \\
\mathbf{C}'_r &= \int_{-1/2}^{+1/2} S_n^2(f)\mathbf{e}(f)\mathbf{e}^T(f)df \\
&+ 2\sum_{k=1}^{K} a_k^2 S_n(f_k)\mathbf{e}(f_k)\mathbf{e}^T(f_k) + \kappa_u \gamma\gamma^T,
\end{aligned}$$

where γ is the $M \times 1$ vector $(\gamma_0, \ldots, \gamma_Q, 0, \ldots, 0)^H$ and where $S_n(f)$ is the power spectral density of n_t. The asymptotic behavior of $\mathbf{s}_T$ and $\mathbf{r}_T$ are directly related by the standard continuity theorem. Therefore:

$$\sqrt{T}\ \ (\mathbf{s}_T - \mathbf{s}(\Theta)) \xrightarrow{\mathcal{L}} \mathcal{N}_c(\mathbf{0}; \mathbf{C}_s(\Theta), \mathbf{C}'_s(\Theta))^1$$

with

$$\mathbf{C}_s(\Theta) \stackrel{\text{def}}{=} \begin{pmatrix} c_r & \mathbf{c}_r^H & \mathbf{c}_r^T \\ \mathbf{c}_r & \mathbf{C}_{r'} & \mathbf{C}'_{r'} \\ \mathbf{c}_r^* & \mathbf{C}'^*_{r'} & \mathbf{C}^*_{r'} \end{pmatrix} \tag{9}$$

and

$$\mathbf{C}'_s(\Theta) \stackrel{\text{def}}{=} \begin{pmatrix} c_r & \mathbf{c}_r^T & \mathbf{c}_r^H \\ \mathbf{c}_r & \mathbf{C}'_{r'} & \mathbf{C}_{r'} \\ \mathbf{c}_r^* & \mathbf{C}^*_{r'} & \mathbf{C}'^*_{r'} \end{pmatrix}$$

where $\mathbf{C}_r$ and $\mathbf{C}'_r$ are partitioned as follows

$$\mathbf{C}_r = \begin{pmatrix} c_r & \mathbf{c}_r^H \\ \mathbf{c}_r & \mathbf{C}_{r'} \end{pmatrix} \text{ and } \mathbf{C}'_r = \begin{pmatrix} c_r & \mathbf{c}_r^T \\ \mathbf{c}_r & \mathbf{C}'_{r'} \end{pmatrix}.$$

By application of theorem 1, the covariance of the asymptotic distribution of the minimum variance second-order frequencies estimator (7) is given by the $K \times K$ "frequency corner" [2] of $(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}$ where $\mathbf{C}_s(\Theta)$ is given by (9).

$$\mathbf{C}_{\Theta_1} = (\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}_{(1:K,1:K)}.$$

Furthermore, because it is proved in [5] that all second-order algorithms are insensitive to the distribution of the additive noise n_t, the third term (which involves the fourth order cumulant κ_u of u_t) in the expressions of $\mathbf{C}_s$ and $\mathbf{C}'_s$ in the expression $(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}$ can be withdrawn.

4.2 Robustness to model errors

Since our approach theoretically applies only for an MA noise of known order, we must consider its robustness to errors model. In fact for any purely nondeterministic random process n_t, $\lim_{k\to\infty}\gamma_k = 0$. So, there exists an integer Q such that $\gamma_k \approx 0$ for $k > Q$. In this situation, the $M \times M$ covariance matrix of x_t becomes $\mathbf{R}(\Theta) + \delta\mathbf{R}$ where $\delta\mathbf{R}$ gathers "small" terms $\gamma_{Q+1}, \ldots, \gamma_{M-1}$. This implies a biased frequency estimator $\Theta_{1,T}$. This asymptotic bias is given by:

$$\mathrm{E}(\Theta_{1,T}) - \Theta_1 = \mathbf{D}_{\mathbf{s}}^{\mathrm{AMV}_1}\delta\mathbf{s} + O(\|\delta\mathbf{s}\|^2) + O(\tfrac{1}{T}),$$

where

$$\delta\mathbf{s} \;=\; (0,..,0,\gamma^*_{Q+1},\ldots,\gamma^*_{M-1},0,..,0,\gamma_{Q+1},..,\gamma_{M-1})^T$$

and $\mathbf{D}_{\mathbf{s}}^{\mathrm{AMV}_1}$ denotes the derivative of the AMV algorithm defined in section 2 which is given by

$$\mathbf{D}_{\mathbf{s}}^{\mathrm{AMV}_1} = \left[(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\right]_{(1:K,:)}$$

(using [6, proof of theorem 1]. And the asymptotic covariance is given by:

$$\begin{aligned}\mathrm{Cov}(\Theta_{1,T}) = \tfrac{1}{T}\mathbf{D}_{\mathbf{s}+\delta\mathbf{s}}^{\mathrm{AMV}_1}\mathbf{C}_{s+\delta s}(\Theta)\left(\mathbf{D}_{\mathbf{s}+\delta\mathbf{s}}^{\mathrm{AMV}_1}\right)^H \\ +O(\|\delta\mathbf{s}\|^4) + O(\tfrac{1}{T^2}) + O(\|\delta\mathbf{s}\|^2)O(\tfrac{1}{T}).\end{aligned}$$

5 Simulations

In this section, we provide numerical illustrations of the performance of the proposed AMV estimator as well as a comparison with the MPD estimator of [1]. We note, that contrary to the AMV approach for which $M \geq Q + K + 1$, the MPD estimator, based on an Hankel matrix built from the "zero triangles" of $\mathbf{R}(\Theta)$, requires the fixed order $M = Q + 2K + 2$.

In the first experiment, we consider the case $K = 2$ ($f_1 = -0.1$, $f_2 = 0.3$), $Q = 1$ ($b_0 = 0.707$, $b_1 = 0.707$) and $T = 2000$ [3] for different SNR. Fig.1 shows the asymptotic variance given by the MPD estimator for $M = 7$ ($\mathrm{Var}[\theta_{1,T}] = \frac{1}{T}\mathbf{D}_{\mathbf{r}}^{\mathrm{MPD}}\mathbf{C}_r(\Theta)(\mathbf{D}_{\mathbf{r}}^{\mathrm{MPD}})^H_{(1,1)}$) and by our estimator for $M = 5$, 7, 9 and 11. We see the

[1] We note that the noncircular complex Gaussian distribution of $\mathbf{s}_T$ is characterized by $\mathbf{C}_s$ only.

[2] We note that the expression $(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}$ where $\mathbf{S}$ would be defined as the derivative of $\mathbf{s}(\Theta)$ w.r.t. Θ_1 is an even lower bound, but because no known second-order algorithm attains this bound, this lower bound has no practical interest.

[3] This value of T is chosen large enough because it is well known (see e.g., [7]) that T has to be very large for the asymptotic expressions to describe the estimated MSE's adequately for large value of M.

improvement of performance of our approach including for a smaller order M.

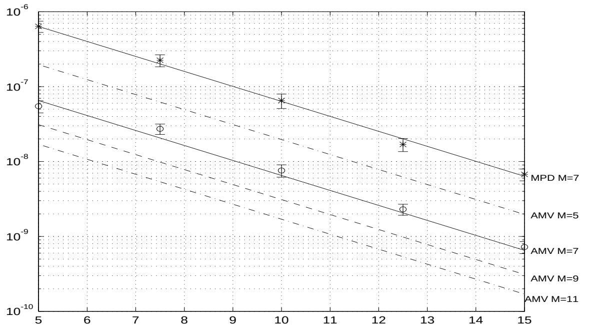

Fig.1 Theoretical and estimated (100 runs with 99% confidence interval) variance $\text{Var}[\theta_{1,T}]$ given by the MDP ($M = 5$) and the AMV estimator (M = 5, 7, 9 and 11) versus SNR.

The second experiment illustrates the influence of the frequency of the complex sinusoid on the performance. We assume $K = 1$, $Q = 1$ ($b_0 = 0.707$, $b_1 = -0.218 - 0.672i$), $M = 4$, $T = 2000$ and SNR = 5 dB. The power spectral density of the MA process is shown in Fig.2a. Fig.2b shows the asymptotic variance given by the AMV estimator as a function of the frequency of the complex sinusoid (1). Fig.2b also plots the asymptotic variance given by the AMV (3) and the MUSIC (2) estimator for a white noise of same power. We see that the MUSIC estimator is not a minimum variance second-order estimator. This figure shows a degradation of the performance when the frequencie f_1 is in the vicinity of the maximum of $S_n(f)$. This result is similar to the performance of the nonlinear least square estimator [8], where the asymptotic variances are proportional to $S_n(f_1)/a_1^2$.

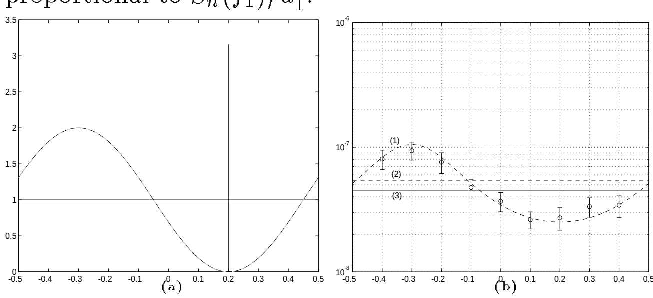

Fig.2a Power density spectra for white (-) and MA (- -) noise.

Fig.2b Theoretical and estimated (100 runs with 99% confidence interval) variance $\text{Var}[\theta_{1,T}]$ given by the AMV estimator for MA noise (1) and by the AMV (3) and MUSIC (2) estimators for white noise versus f_1.

Finally, the influence of an error of model is examined. We consider in this experiment, an AR model of order 1 with $K = 1$, ($f_1 = 0.2$), $M = 7$, SNR = 5 dB, $T = 2000$. Fig.3 plots the asymptotic bias and mean square error given by the MPD and AMV estimators when the order of the MA model is falsely detected to be $Q = 2$ as a function of the AR coefficient. We see that these two estimators are robust to this undermodeling. Furthermore, the AMV estimator continues to outperform the MPD estimator.

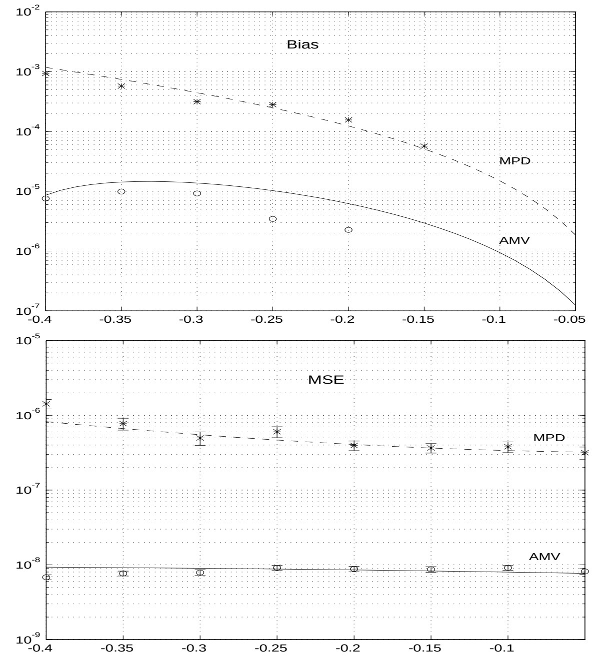

Fig.3 Theoretical and estimated (100 runs) asymptotic bias and MSE of the AMV and MPD estimator as a function of the AR coefficient.

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