GENERATION OF ULTRASPHERICAL WINDOW FUNCTIONS

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ABSTRACT

Two methods for the computation of the coefficients of the ultraspherical window are presented. The first corresponds to a concise exposition of Streit's method. The second is a new method that involves equating an ultraspherical window's frequency-domain representation to a Fourier series from which the coefficients are readily found. The two methods produce exactly the same results given the same window parameters. Also outlined is a method for the selection of an ultraspherical window's parameters based on the Dolph-Chebyshev window. This method highlights the flexibility of the ultraspherical window.

1 INTRODUCTION

Window functions are used to reduce Gibbs' oscillations and find a wide array of applications from power spectral estimation to digital filter design. With a large number of applications available, window flexibility becomes a key concern. One such flexible window function is the ultraspherical window which has three independent parameters for controlling it's properties.

A substantial amount of literature exists involving classical window functions and their merits. Many of the available windows were obtained by exploiting certain characteristics of well known polynomials where each window function best satisfies a particular criterion. For instance, the Dolph-Chebyshev window [1] produces a minimum main-lobe width for a specified maximum side-lobe level whereas the Kasier [2] and Saramäki [3] windows achieve close approximations to the discrete prolate functions which have a maximum energy concentration in the main lobe relative to that of the side lobes.

Not very long ago, Streit [4] explored the use of ultraspherical polynomials[1] to produce various field patterns for symmetric equally spaced broadside antenna arrays which may also be known as ultraspherical windows. The use of ultraspherical polynomials introduces another degree of freedom relative to typical adjustable window designs[2] such as the Dolph-Chebyshev, Kaiser, and Saramäki windows. In fact, Streit has shown that the Dolph-Chebyshev and Kaiser windows are particular cases of the ultraspherical window. With such flexibility, and the ability of the ultraspherical window to include many classical windows as specific cases, it is worthwhile to explore its use further.

After Streit's work, Soltis [5]-[6] and Saèd et al. [7] used ultraspherical polynomials to further investigate Streit's antenna array approximations and used them in wavelet analysis. Deczky [8] used ultraspherical windows for nonrecursive digital filters.

In this paper, two methods for the computation of the coefficients of the ultraspherical window are presented. The first is a concise exposition of Striet's method and the second is a new method that involves equating an ultraspherical window's frequency-domain representation to a Fourier series. In addition, a method is outlined for the selection of an ultraspherical window's parameters based on the Dolph-Chebyshev window. This method highlights the flexibility of the ultraspherical window.

2 ULTRASPHERICAL WINDOW COEFFICIENTS

Streit's equations for an ultraspherical window can be expressed as

$$w(nT) = b_{M-|n|,2M}(x_\mu) \quad \text{for } n \leq |M| \tag{1}$$

where μ, x_μ, and M are independent parameters. The window length is $N = 2M+1$, $w(nT) = w(-nT)$, and the normalized ultraspherical window is obtained as $\widehat{w}(nT) = w(nT)/w(0)$. The coefficients $b_{M-|n|,2M}(x_\mu)$ are found through the equation

$$\begin{aligned} b_{k,n}(x_\mu) &= \frac{2\mu x_\mu^n}{n-k}\binom{\mu+n-k-1}{n-k-1} \\ &\quad \cdot \sum_{m=0}^{k}\binom{\mu+k-1}{k-m}\binom{n-k}{m}A^m \end{aligned} \tag{2}$$

[1] Also referred to as Gegenbauer polynomials for Gegenbauer's related work on their properties.

[2] Adjustable windows typically have two adjustable parameters, namely (1) the window length which alters the main-lobe width and (2) a parameter which alters side-lobe height.

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n}{k}\cdot\frac{n-1}{k-1}\cdots\frac{n-k+2}{2}\cdot\frac{n-k+1}{1} \qquad (3)$$

and $A = 1 - x_\mu^{-2}$ so that $0 < A < 1$ when $x_\mu > 1$.

A second method for the computation of the window coefficients involves equating an ultraspherical window's frequency-domain representation to a Fourier series. To start, we take a lead from Stegen [9] where he notes that a sum

$$F(x) = \sum_{m=0}^{r} (a_m \cos mx + b_m \sin mx) \qquad (4)$$

can be found that furnishes the best possible representation of a function $u(x)$ that takes the values u_0, u_1, u_2, ..., u_{n-1}, when x takes the values 0, $2\pi/n$, $4\pi/n$, ..., $2(n-1)\pi/n$, respectively, where $n \geq 2r+1$. The coefficients in Eq. (4) are given by

$$a_0 = \frac{1}{n}\sum_{k=0}^{n-1} u_k \qquad (5)$$

$$a_m = \frac{2}{n}\sum_{k=0}^{n-1} u_k \cos\frac{2k\pi m}{n} \qquad (6)$$

and

$$b_m = \frac{2}{n}\sum_{k=0}^{n-1} u_k \sin\frac{2k\pi m}{n}. \qquad (7)$$

If we set $r = M$ and $n = 2r+1$, the values $u_k = u(x_k)$ in Eqs. (5), (6), and (7) are found by setting

$$u(x) = C_{2M}^{\mu}\left(x_\mu \cos\frac{x}{2}\right) \qquad (8)$$

and subsequently finding $u(x_s)$ at $2M+1$ points distributed over x given by

$$x_s = \frac{2\pi}{2M+1}s \quad \text{for } s = 0, 1, ..., 2M \qquad (9)$$

where

$$u_s = u(x_s) = C_{2M}^{\mu}\left(x_\mu \cos\frac{\pi s}{2M+1}\right). \qquad (10)$$

With $b_m = 0$, the expressions for coefficients a_0 and a_m become

$$\begin{aligned} a_0 &= \frac{1}{2M+1}\sum_{s=0}^{2M} u_s \\ &= \frac{1}{2M+1}\left(u_0 + \sum_{s=1}^{M} u_s + \sum_{s=M+1}^{2M} u_s\right) \end{aligned} \qquad (11)$$

and

$$\begin{aligned} a_m &= \frac{2}{2M+1}\sum_{s=0}^{2M} u_s \cos\frac{2s\pi m}{2M+1} \\ &= \frac{2}{2M+1}\left(u_0 + \sum_{s=1}^{M} u_s \cos\frac{2s\pi m}{2M+1}\right. \\ &\quad \left. + \sum_{s=M+1}^{2M} u_s \cos\frac{2s\pi m}{2M+1}\right). \end{aligned} \qquad (12)$$

Now in an effort to simplify the above expressions, we note that $C_{2M}^{\mu}(x_\mu \cos x/2)$ has a degree $2M$ in $X = x_\mu \cos x/2$. As such, it is an even function of X implying that $C_{2M}^{\mu}(X) = C_{2M}^{\mu}(-X)$. Using this property, Eqs. (11) and (12) yield

$$\begin{aligned} a_0 &= \frac{1}{2M+1}\left[C_{2M}^{\mu}(x_\mu)\right. \\ &\quad \left. +2\sum_{s=1}^{M} C_{2M}^{\mu}\left(x_\mu \cos\frac{\pi s}{2M+1}\right)\right] \end{aligned} \qquad (13)$$

and

$$\begin{aligned} a_m &= \frac{2}{2M+1}\left[C_{2M}^{\mu}(x_\mu)\right. \\ &\quad +2\sum_{s=1}^{M} C_{2M}^{\mu}\left(x_\mu \cos\frac{\pi s}{2M+1}\right) \\ &\quad \left. \cdot \cos\frac{2\pi s m}{2M+1}\right]. \end{aligned} \qquad (14)$$

We can now express the window coefficients for the ultraspherical window as

$$\begin{aligned} w(nT) &= \frac{1}{2M+1}\left[C_{2M}^{\mu}(x_\mu)\right. \\ &\quad +2\sum_{s=1}^{M} C_{2M}^{\mu}\left(x_\mu \cos\frac{\pi s}{2M+1}\right) \\ &\quad \left. \cdot \cos\frac{2\pi s n}{2M+1}\right] \quad \text{for } |n| \leq M. \end{aligned} \qquad (15)$$

The normalized ultraspherical window is obtained as $\widehat{w}(nT) = w(nT)/w(0)$. This method of computation of the ultraspherical window coefficients produces exactly the same results as Eq. (1) given the same set of values for the independent parameters μ, x_μ, and M.

3 CLOSED-FORM GENERATION OF ULTRASPHERICAL POLYNOMIALS

When using Eq. (15) for the calculation of the ultraspherical window coefficients, closed-form calculation of $C_{2M}^{\mu}(x_\mu \cos x/2)$ is required. Given μ, x_μ, M, and x, such a closed-form calculation may be preformed using a standard recurrence relationship for the ultraspherical polynomial found in Abramowitz [10] given by

$$\begin{aligned} C_n^{\mu}(x) &= \frac{1}{n}\left[2x(n+\mu-1)C_{n-1}^{\mu}(x)\right. \\ &\quad \left. -(n+2\mu-2)C_{n-2}^{\mu}(x)\right] \end{aligned} \qquad (16)$$

for $n = 2, 3, ..., 2M$, where $C_0^\mu(x) = 1$ and $C_1^\mu(x) = 2\mu x$.

For the case where $\mu = 0$, the ultraspherical window becomes the Dolph-Chebyshev window and the recurrence relationship for the Chebyshev polynomial $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ with $T_0(x) = 1$ and $T_1(x) = x$ must be used.

4 SELECTION OF INDEPENDENT PARAMETERS

The two methods presented for the computation of the ultraspherical window coefficients require the independent parameters μ, x_μ, and M. In this section, we briefly explore the calculation of these parameters using known design equations for the Dolph-Chebyshev window.

The frequency-domain representation of the Dolph-Chebyshev window is known for having (1) all side lobes at the same amplitude and (2) a minimum main-lobe width u_0 for a specified maximum side-lobe level. Typically in Dolph-Chebyshev window designs the independent parameters, x_0 and M, are calculated to yield a specified ripple ratio r in the window's frequency-domain representation given by

$$r = \frac{\text{maximum side-lobe amplitude}}{\text{main-lobe amplitude}}. \qquad (17)$$

If the desired ripple ratio is S dB, then r is found as

$$r = 10^{-S/20}. \qquad (18)$$

The calculation of x_0 is then preformed using the relation

$$x_0 = \cosh\left(\frac{1}{2M}\cosh^{-1} 1/r\right). \qquad (19)$$

In the design of the ultraspherical window we must choose a specific value for μ. If we set $u_0 = u_\mu$ where u_μ is the main-lobe width of the ultraspherical window, then x_μ is found as suggested by Streit [4] by using the relation

$$x_\mu = x_0 x_{2M}^{(\mu)} \sec\left(\frac{\pi}{4M}\right) \qquad (20)$$

where $x_{2M}^{(\mu)}$ is the largest zero of the ultraspherical polynomial $C_{2M}^\mu(x)$. As there is no explicit formula for the zeros of the ultraspherical polynomial, $x_{2M}^{(\mu)}$ can be found by applying the Newton-Raphson iteration

$$y_{k+1} = y_k - \frac{C_{2M}^\mu(y_k)}{2\mu C_{2M-1}^{\mu+1}(y_k)} \quad \text{for } k = 1, 2, ... \qquad (21)$$

assuming the value $y_1 = x_{2M}^{(0)} = \cos(\pi/4M)$ for the first iteration. The procedures explained in the previous section can be used to calculate specific values of $C_n^\mu(x)$.

An important implication of using Eq. (20) is that it sets $u_\mu = u_0$ for different values of μ, in effect forcing the first null to remain in the same position. Varying μ thus allows control over the placement of the other nulls in the window which enables one to achieve a variety of side-lobe patterns.

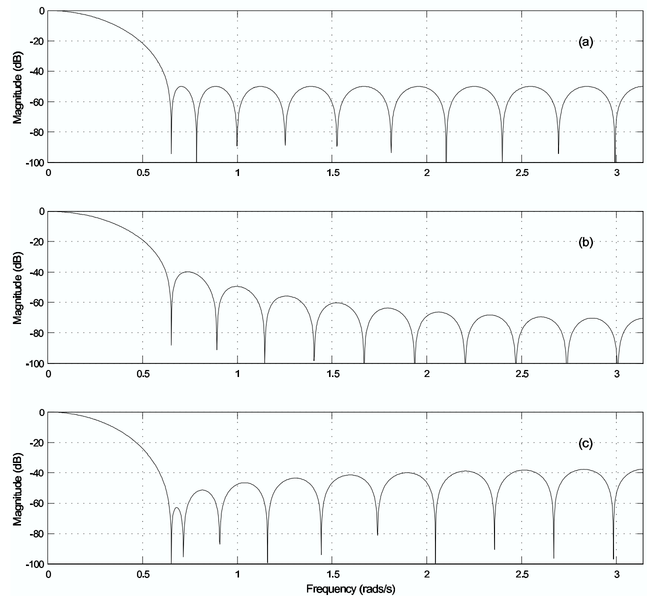

Figure 1: Frequency-domain representation of ultraspherical windows: (a) $\mu = 0$ (Dolph-Chebyshev case), (b) $\mu = 3.0$, (c) $\mu = -1.1$.

5 RESULTS

If we hold x_0 and M constant and vary μ, the side-lobe pattern of the window may be adjusted while keeping a constant main-lobe width, i.e., $u_\mu = u_0$. This becomes apparent if we set $M = 10$, $S = 50$ dB and evaluate the window for $\mu = 0$, 3, and -1.1. The frequency-response representations for the three cases are illustrated in Figs. 1a to 1c and the corresponding coefficients are given in Table 1.

A closer look around the first null for all three μ values is shown in Fig. 2. As expected, the main-lobe width remains constant for the three values of μ at $u_0 = u_3 = u_{-1.1} = 2*0.6524 = 1.3048$ but the side-lobe pattern for each window changes substantially. If we use the Dolph-Chebyshev window, i.e., for $\mu = 0$, as a reference for $\mu = 3$ and -1.1, the peak of the first side lobe is 10.16 dB greater and 12.82 dB lower, respectively. At the same time, the furthest side-lobe peak becomes 20.64 dB lower and 12.29 dB greater then that in their Dolph-Chebyshev counterpart.

To characterize the relationship between μ and the side-lobe pattern, we introduce the ratio

$$R_{side} = \log_{10}(a_1/a_2) \qquad (22)$$

where a_1 is the height of the side lobe nearest to the main lobe and a_2 is the height of the side lobe furthest from the main lobe. A plot of R_{side} vs. μ for various values of M is shown in Fig. 3. We observe that if $\mu > 0$ then $a_1 > a_2$ and if $\mu < 0$ then $a_1 < a_2$. It is also important to note that R_{side} depends on μ and M.

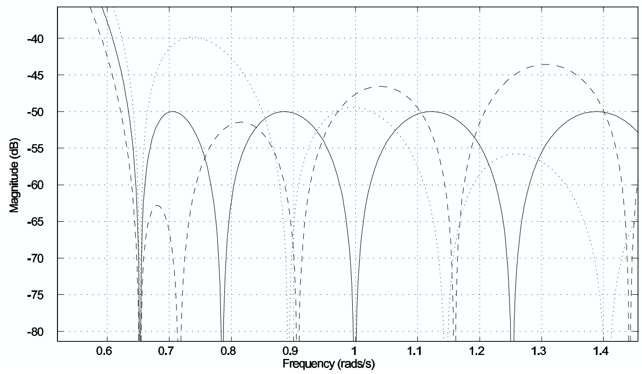

Figure 2: Comparison of window functions with $S = 50$ dB and $M = 10$ for the cases where $\mu = 0$ (solid line), $\mu = 3$ (dotted line), and $\mu = -1.1$ (dashed line).

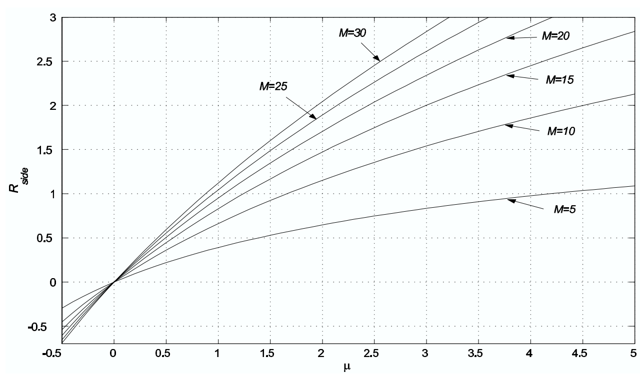

Figure 3: Relationship between side-lobe decay ratio R_{side} and μ for various values of M.

This implies that for a given value of μ, different R_{side} values are produced for different M values.

6 CONCLUSIONS

Two methods for the computation of the coefficients of the ultraspherical window were presented. The two methods use the independent parameters μ, x_μ, and M to alter a given window's properties and give exactly the same results.

A procedure for calculating the independent parameters given a desired frequency-domain representation of the window was also outlined. This procedure is based on using known equations for the Dolph-Chebyshev window as a reference for the ultraspherical window.

Table 1: Ultraspherical window coefficients for $\mu = 0$, 3, and -1.1 with $M = 10$ and $S = 50$ dB.

$w(nT)$	$\mu = 0$	$\mu = 3$	$\mu = -1.1$
$w(0)$	1.0000	1.0000	1.0000
$w(1)$	0.9760	0.9745	0.9777
$w(2)$	0.9069	0.9010	0.9134
$w(3)$	0.8010	0.7883	0.8144
$w(4)$	0.6704	0.6496	0.6913
$w(5)$	0.5293	0.5002	0.5566
$w(6)$	0.3914	0.3553	0.4229
$w(7)$	0.2680	0.2281	0.3007
$w(8)$	0.1669	0.1273	0.1979
$w(9)$	0.0914	0.0571	0.1017
$w(10)$	0.0470	0.0164	0.0982

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