THE LIFTING THEOREM AND THE DESIGN OF BIORTHOGNAL FILTER BANKS

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ABSTRACT

In this paper, a simple design method is proposed for the construction of biorthogonal filter banks with perfect reconstruction property. Starting from an analysis filter bank that meets certain design specifications regarding bandwidth and energy concentration, a complete biorthogonal filter bank system is constructed. Then, through applying the lifting scheme to the other analysis filter bank, its response can be greatly improved to meet the desired design specifications without impairing either the perfect construction property or the other analysis filter bank. Illustrative example is given to demonstrate the simplicity and efficiency of the proposed design approach.

1. Introduction:

Recently [1-3], interest has been given to the use of biorthogonal wavelets in signal processing. In orthogonal wavelet decomposition, the single orthogonal wavelet function $\psi(t)$ plays two very important roles in signal analysis. First, it is used to construct orthonormal basis $\psi_{j,k}(t)$ that is used as analyzing wavelets to provide certain time-frequency resolution. Second, it is used to construct the waveform at various octaves of the signal. This means that the analysis and synthesis wavelets are the same. Furthermore, it is known that orthonormal scaling functions and wavelets have poor time scale localization. Most of available QMF designs [4], are based on orthogonal wavelets with its drawback of having a single scaling function to meet the perfect construction property as well as selective linear phase response for each of the analysis filter bank to meet energy concentration in these banks. The use of biorthogonal systems rather than orthogonal system gives us a lot of freedom. Here, we have a pair of wavelets $\psi(t),\tilde{\psi}(t)$, *(called duals of each other),* that are used to share the load; one as an analyzing wavelet and the other as synthesizing wavelet.

In [5], a method was proposed to design biorthogonal filter banks. However, it was shown that its overall response is not an exact PR one. In [6], as interest was focused on image compression, the criterion for the design of biorthogonal filter banks was to ensure the smoothness of the constructed wavelets.

In this paper, a simple method is described for the construction of a biorthognal basis for any specific application. Next, the lifting scheme mathematically described in [7-8], has been used to design a biorthogonal perfect construction filter banks. In this respect, for a specific low pass analysis filter bank that meets certain design specifications, the corresponding high pass bank that satisfies the biorthogonality requirements is hardly a good high pass. Next, the lifting scheme is applied to make the high pass analysis filter approximates the required ideal high pass specifications, without impairing the overall PR property, the linear phase property of each of the analysis/synthesis filter banks, or the low pass analysis filter bank. Illustrative example is given to show the effectiveness of this approach..

2. Biorthogonal System:

Let the scaling function $\phi(t)$ has a dual scaling function $\tilde{\phi}(t)$ and the associated wavelet function $\psi(t)$ has the corresponding dual function $\tilde{\psi}(t)$. Then, to have a biorthogonal system, the following conditions must be satisfied

$$\begin{aligned} &\int \phi(t)\overline{\tilde{\phi}(t-k)}\,dt=\delta(k) && \int \psi(t)\overline{\tilde{\phi}(t-k)}\,dt=0 \\ &\int \psi(t)\overline{\tilde{\psi}(t-k)}\,dt=\delta(k) && \int \phi(t)\overline{\tilde{\psi}(t-k)}\,dt=0 \end{aligned} \qquad (1)$$

In terms of the two scale relations of the scaling and wavelet functions and their dual, defined as

$$\begin{aligned} \phi(t)&=\sum_{k=0}^{N} h_o(k)\phi(2t-k) & \tilde{\phi}(t)&=\sum_{k=0}^{N} g_o(k)\tilde{\phi}(2t-k) \\ \psi(t)&=\sum_{k=0}^{N} h_1(k)\phi(2t-k) & \tilde{\psi}(t)&=\sum_{k=0}^{N} g_1(k)\tilde{\phi}(2t-k) \end{aligned} \qquad (2)$$

where the order of $(H_o(z), G_o(z), H_1(z), G_1(z))$, is taken to be equal to N . Eq.(1) is equivalent to

$$\begin{aligned} &H_o(z)G_o(z^{-1})+H_o(-z)G_o(-z^{-1})=1 && (a) \\ &H_o(z)G_1(z^{-1})+H_o(-z)G_1(-z^{-1})=0 && (b) \\ &H_1(z)G_1(z^{-1})+H_1(-z)G_1(-z^{-1})=1 && (c) \\ &H_1(z)G_o(z^{-1})+H_1(-z)G_o(-z^{-1})=0 && (d) \end{aligned} \qquad (3)$$

where

$$H_o(z)=\frac{1}{2}\sum_{k=o}^{N} h_o(k)\,z^{-k},\quad G_o(z)=\frac{1}{2}\sum_{k=o}^{N} g_o(k)z^{-k}$$

$$H_1(z)=\frac{1}{2}\sum_{k=o}^{N} h_1(k)z^{-k}\ ,\quad H_o(z)=\frac{1}{2}\sum_{k=o}^{N} g_1(k)z^{-k}\quad z=e^{j\frac{\omega}{2}} \tag{4}$$

At this point, it is worth noting that orthogonal wavelet is a special case of biorthogonal system in that the dual functions $\tilde{\phi}(t),\tilde{\psi}(t)$ lie in the same plane as that of the functions $\phi(t),\psi(t)$. As a result, in orthogonal wavelets, Eq.(2) is satisfied with the conditions $g_0(n)= h_0(n)$, $g_1(n)=h_1(n)$.

To relate the biorthogonal filter banks to the perfect reconstruction filter banks [4], it is well known that to have PR system, the following relations must be satisfied

$$\begin{aligned} H_o(z)F_o(z)+H_1(z)F_1(z) &= c\,z^{-K} \\ H_o(-z)F_o(z)+H_1(-z)F_1(z) &= 0 \end{aligned} \tag{5}$$

where $\{H_0(z), H_1(z)\}$ and $\{F_0(z), F_1(z)\}$ are the low and high pass decomposition and reconstruction filter banks, respectively. Comparison with the biorthogonal relations of eq.(3), yields

$$F_o(z)=z^{-K}G_o(z^{-1})\quad ,\quad F_1(z)=H_o(-z) \tag{6}$$

3. The Lifting Principle:

Investigation of eq.(3), reveals the following remarks:

1. If in the biorthogonal system $\{H_o(z), H_1(z), G_0(z), G_1(z)\}$, the functions $[H_1(z),G_0(z)]$ are kept unchanged while one modifies the other parameters to $\{H_o^n(z),G_1^n(z)\}$ in accordance with

$$\begin{aligned} H_o^n(z) &= H_o(z)+W(z^{-2})H_1(z) \\ G_1^n(z) &= G_1(z)-W(z^2)\,G_0(z) \end{aligned} \tag{7}$$

then, the system $\{H_o^n(z),H_1(z),G_o(z),G_1^n(z)\}$ is still biorthogonal for any arbitrary $W(z)$. Direct substitution of $H_0^n(z)$, $G_1^n(z)$ in place of $H_0(z),G_1(z)$ in Eq.(3), reveals that the biorthogonal relations are satisfied as long as the original system $H_o(z)$, $H_1(z)$, $G_o(z)$, $G_1(z)\}$ is biorthogonal. Alternatively, if in the biorthogonal system $\{H_o(z), H_1(z), G_o(z), G_1(z)\}$, $[H_o(z),G_1(z)]$ are kept unchanged while the other parameters to $\{H_1^n(z),G_o^n(z)\}$ are modified to

$$\begin{aligned} H_1^n(z) &= H_1(z)+W(z^{-2})H_0(z) \\ G_0^n(z) &= G_o(z)-W(z^2)\,G_1(z) \end{aligned} \tag{8}$$

Then, the system $\{H_o(z),H_1^n(z),G_o^n(z),G_1(z)\}$ is still biorthogona for any arbitrary choice of $W(z)$.

2. As a result of modifying the biorthogonal system $\{H_o(z), H_1(z), G_o(z), G_1(z)\}$, to the modified system $\{H_o(z),H_1^n(z),G_o^n(z),G_1(z)\}$, the new scaling and wavelet functions and their duals are

$$\begin{aligned} \varphi(t) &= \sum_{k=o}^{N} h_o(k)\,\phi(2t-k) \\ \tilde{\varphi}_n(t) &= \sum_{k=o}^{N} g_o^n(k)\,\tilde{\phi}(2t-k) \\ \psi_n(t) &= \sum_{k=o}^{N} h_1^n(k)\,\phi(2t-k) \\ \tilde{\psi}_n(t) &= \sum_{k=o}^{N} g_1(k)\,\tilde{\phi}(2t-k) \end{aligned} \tag{9}$$

It is thus clear that, the scaling function $\phi(t)$ is unchanged, whereas the dual scaling function $\tilde{\phi}(t)$ and the wavelet function $\psi(t)$ has been changed as a result of modifying $\{H_1^n(z),G_o^n(z)\}$, and subsequently the dual wavelet function changes as a result of changing the dual scaling function. This is a very important feature of applying the lifting principl to a biorthogonal system.

Thus, we have the following important theorem, known as the Lifting theorem:

<u>Theorem:</u> The Lifting Theorem

Given a biorthogonal system $\{H_o(z), H_1(z), G_o(z), G_1(z)\}$, then the system $\{H_o^n(z),H_1(z),G_o(z),G_1^n(z)\}$ forms a new biorthogonal system if $\{H_o^n(z),G_1^n(z)\}$ are chosen according to Eq.(7). Alternatively, if $\{H_1^n(z),G_o^n(z)\}$ are allowed to vary in accordance of Eq.(8)., then the resulting system is still biorthogonal , for any arbitrary choice of $W(z)$.

4. Generation of a Biorthogonal system for a specified symmetrical $H_o(z)$:

For a specified symmetrical $H_0(z)$,*(i.e. with linear phase)* of degree $N = 2m+1$, scaled such that $H_0(1)=1$, one can construct a biorthogonal system, as follows:

As $H_0(z)$ is symmetric, it turns out that the coefficients of the biorthogonal system must be symmetric. As a result, the coefficients of $G_0(z)$ are obtained through reducing to zero even power of z, except the zero power term of Eq.(3-a). This results in the following system of $(m+1)$ linear equations in $(m+1)$ unknowns:

$$B\,g_u=y, g_u=[\,g_{0,0}\,g_{1,0}\ldots g_{m,0}\,]^t\ , y=[\,0\,0..0\,0.5\,]^t \tag{10}$$
(12)

and the matrix B can be derived from the coefficients of $H_0(z)$, [9]. The solution of Eq.(1)), yields the vector g_u. Consequently, g_0 = [g_u ; flipud(g_u)]. Once $G_0(z)$ is determined, $H_1(z)$ and $G_1(z)$ that satisfy Eq.(3), can be determined from the relations

$$H_1(z)=z^{-N}G_0(-z^{-1})\ ,\ G_1(z)=z^{-N}H_0(-z^{-1}) \tag{11}$$

In [9], a complete construction procedure is described for a general $H_0(z)$. It remains to determine $W(z)$. In this paper, these elements are determined to ensure selective filter banks with perfect reconstruction PR property, as is described in the following section.

5. Design of Biorthogonal Filter Banks:

The proposed approach to design biorthogonal-based filter banks, proceeds as follows:

1- Specify the linear phase function $H_0(z)$, that suits your design specifications ftom the point of view of energy concentration, allowed numbr of bits,..etc.

2- Determine the rest of the biorthogonal system $H_1(z)$, $G_0(z)$ *and* $G_1(z)$ as described in sec.(4), as well as the reconstruction filter banks $F_0(z)$, $F_1(z)$ of Eq.(6).

3- Apply the lifting theorem of Eq.(8) to $H_1(z)$,to adjust its response to be a good high pass filter. *nature* Note that $W(z^{-2})$ in this case, must be anti-symmetrical to maintain the linear phase as well as the high pass of $H_1^n(z)$. Its parameters are obtained through optimizing the response of the resulting $H_1^n(z)$.

4- As a result of optimizing $W(z^{-2})$, the updated reconstruction filte banks will be

$$F_0^n(z) = F_0(z) - W(z^{-2})\,H_0(-z)$$
$$F_1^n(z) = F_1(z) \qquad (12)$$

It can be easily verified that, the resulting biorthogonal filter bank system, is indeed PR with no aliasing, i.e.

$$T(z) = H_0(z)F_0^n(z) + H_1^n(z)F_1^n(z) \quad \equiv z^{-K}$$
$$A(z) = H_0(-z)F_0^n(z) + H_1^n(-z)F_1^n(z) \equiv 0 \qquad (13)$$

This feature is to be compared with earlier biorthogonal designs [5], where their overall $T(z)$ deviates subtantially around $\frac{\pi}{2}$.

6. Illustrative Example:

Ex.: In this example, we will design a 23[th].degree FIR filter $H_o(z)$, using Remez exchange algorithm with frequency edges [0 0.45 0.6 1], and magnitude [1 1 0 0].

Fig.(1-a), shows the amplitude response of the designed filter together with amplitude response of $H_0(z)$ and the corresponding $H_1(z)$ obtained from the biorthogonality conditions of Eq.(3). Next, we apply the lifting principle to improve $H_1(z)$ while keeping $H_0(z)$ invariant. The parameters of $W(z)$ are determined such that they minimize the amplitude deviations of the resulting $H_1^n(z)$ over the specified frequency specifications. Fig.(1-b), shows the effect of applying the lifting theorem with 4,5 variables on the response of $H_1^n(z)$, while fig.(2) shows their expanded stop and pass bands, respectively. $T(z)$ of the resulting filter bank is exactly an ideal delay of 31 and 33 samples, respectively, while A(z) is exactly zero.

7. Conclusions:

A simple method is presented for the design and construction of biorthogonal filter bank system, with perfect construction property and zero aliasing. The main feature of this design lies in the fact that the lifting principle can be easily incorporated in the resulting design. This means that one can choose one of the analysis filter banks freely without any binding relation to the other analysis filter bank. The parameters of the lifting functions can be chosen to satisfy any different design schemes, such as to ensure scaling function regularity,[8].

8. References:

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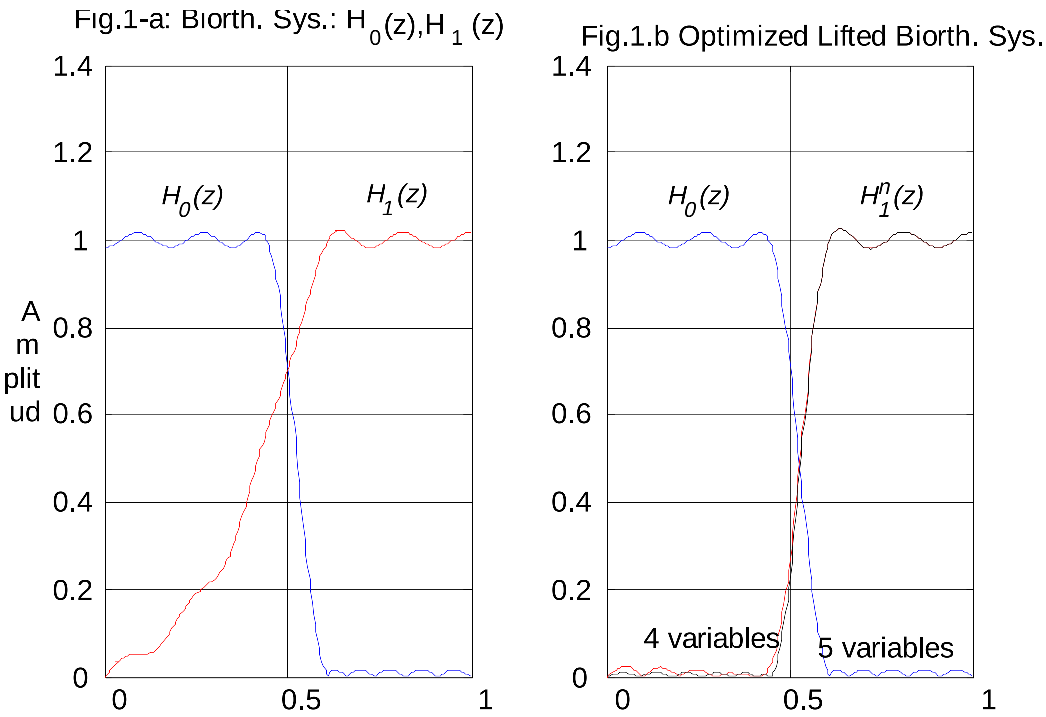

Fig.1 a: Biorthogonal analysis filter banks before lifting
b: Optimized lifted biorthogonal analysis filter banks

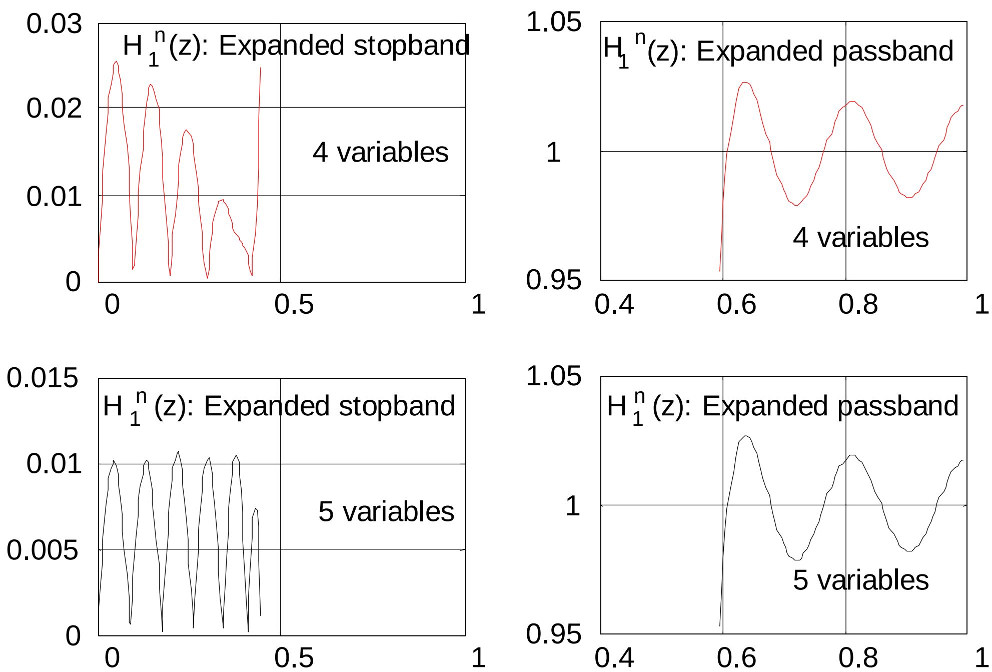

Fig. 2. Expanded pass and stop band behavior of the optimized lifted $H_1^n(z)$ for cases of $W(z^{-2})$ having 4 and 5 variables, respectively