CIRCLE FITTING BY ITERATIVE INVERSION

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ABSTRACT

This paper recalls a relatively unknown method of circle fitting (J. A. Brandon and A. Cowley, "A weighted least squares method for circle fitting to frequency response data," *Journal of Sound and Vibration*, vol. 83, no. 3, pp. 419–424, 1983) which has been derived using a classical geometric result. We propose a modification of this technique by re-weighting the data, iterating the procedure and choosing at every step as the new inversion point the one diametrically opposite to the previous.

1 INTRODUCTION

Many problems ask for the fitting of a circle with a set of noisy observations. Such situations often occur where a complex bilinear transformation is involved. The Hough transform is not suitable as in almost all cases the size of the data is not sufficiently large compared to the numbers of unknowns. Another widely used method to fit a curve through scattered points in a plane is the orthogonal distance regression (ODR). This determines the curve which minimizes the sum of square of distances from each data point to the closest point on the curve. ODR gives good results, but it is quite computationally expensive [5]. The simple method of Kasa [3] gives biased estimates of the circle center unless the data are symmetrically distributed around the circumference of the circle.

The approach presented in this paper is based on the property of an inversion transformation to map a circle into a straight line, if the circle passes through the pole of the inversion [4]. The method was originally introduced in [1], but it is relatively unknown. Our contribution consists in modifying the algorithm by re-weighting the data, iterating the procedure and choosing at every step as the new inversion point the one diametrically opposite to the previous inversion point. It follows that our objective in this paper is essentially twofold: to recall the circle fitting method by geometric inversion and to present the iterative algorithm. Nevertheless, with such a type of technique, the extensions can be readily obtained and furthermore, it should be worth more investigation. However, due to space limitations, other aspects will be the goal of a future work [6].

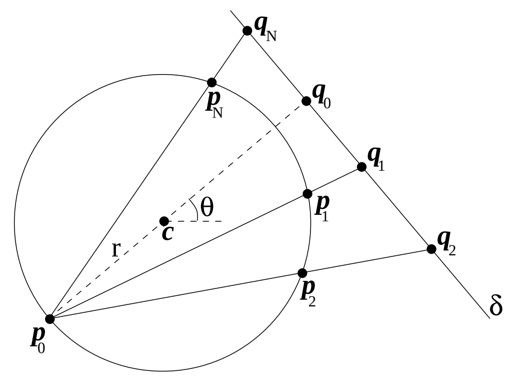


Figure 1: The inversion transformation.

2 PREVIOUS RESULTS

Let $\{\mathbf{p}_i = [p_i(x)\ p_i(y)]^T | i = \overline{0,N}\}$ denote a set of $N+1$ different points on a circle. The inversion transform with pole $\mathbf{p}_0$ and coefficient k maps the points $\{\mathbf{p}_i \mid i = \overline{1,N}\}$ into their corresponding images $\{\mathbf{q}_i \mid i = \overline{1,N}\}$ as follows

$$\mathbf{q}_i = \mathbf{p}_0 + k^2(\mathbf{p}_i - \mathbf{p}_0)/(d_i^2), \quad \forall\ i = 1, 2, \ldots, N, \quad (1)$$

where d_i is the distance between $\mathbf{p}_i$ and the pole $\mathbf{p}_0$. Because the circle passes through the pole $\mathbf{p}_0$, all the image points $\{\mathbf{q}_i | i = \overline{1,N}\}$ lie on a certain line δ as shown in Fig 1. The line δ together with the inversion parameters (the pole and the coefficient k) uniquely determine the circle. Thus the problem of fitting a circle with a given set of points $\mathbf{p}_i$ has been reduced in [1] to the more simple problem of fitting a line to another set of points $\{\mathbf{q}_i | i = \overline{1,N}\}$. Once the orientation of the line δ and the distance from the pole $\mathbf{p}_0$ to the closest point on the line $\mathbf{q}_0$ are found, the center $\mathbf{c}$ and the radius r of the circle can be determined by noting that $r = k^2/(2d_0)$, $\mathbf{c} = \mathbf{p}_0 + r \cdot \mathbf{u}_\theta$, where d_0 is the distance between $\mathbf{p}_0$ and $\mathbf{q}_0$ and, $\mathbf{u}_\theta = [\cos\theta \sin\theta]^T$ is the unit vector orthogonal to the line δ as shown in Fig 1. In other words, θ is the direction which minimizes the variance of the projections of q_i [2].

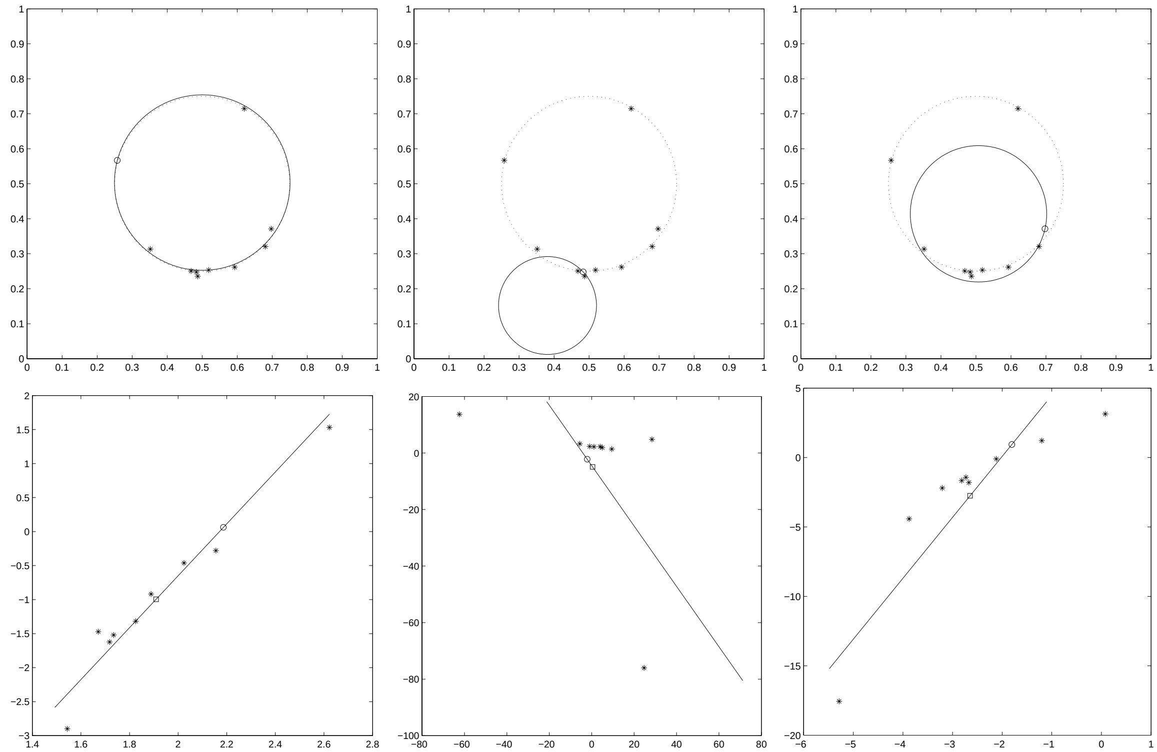
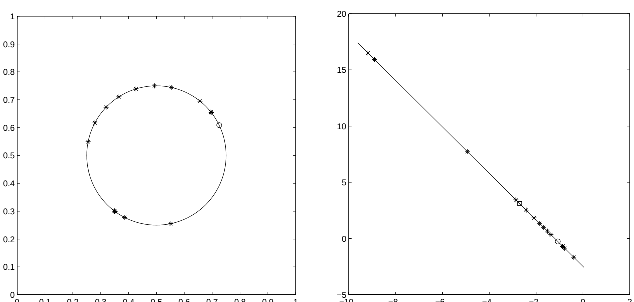


Figure 3: Examples of circle estimation based on three inversion transformations with different poles when the observed point set is affected by noise.



(a) (b)

Figure 2: Example of circle fitting with a set of points (a), and the transform points (b).

3 THE PROPOSED METHOD

Let $\sigma^2(\alpha)$ denote the variance of the projections of $\mathbf{q}_i$ points onto the direction given by the unity vector $\mathbf{u}_\alpha = [\cos\alpha \quad \sin\alpha]^T$

$$\sigma^2(\alpha) = \sum_{i=1}^{N} [\mathbf{u}_\alpha^T(\mathbf{q}_i - \bar{\mathbf{q}})]^2, \text{ where } \bar{\mathbf{q}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{q}_i. \quad (2)$$

A Least Square estimator of θ is obtained as in [7] by minimizing (2) and it can be expressed in terms of central moments μ_{mn} of the set $\{\mathbf{q}_i | i = \overline{1, N}\}$ as follows

$$\theta = \frac{1}{2}\arctan\left(\frac{2\mu_{11}}{\mu_{20} - \mu_{02}}\right) + \text{sign}\left[\text{sign}(\mu_{11}) - 0.5\right] \times \times \{1 - \text{sign}\left[1 + \text{sign}(\mu_{20} - \mu_{02})\right]\} \frac{\pi}{2} + \frac{\pi}{2},$$
$$\mu_{mn} = \frac{1}{N}\sum_{i=1}^{N}[q_i(x) - \bar{q}(x)]^m[q_i(y) - \bar{q}(y)]^n. \quad (3)$$

Example 1. *Circle fitting for an ideal case*

We consider the situation when there is no noise present and all the given points are on the circle. Such a circle is shown in Fig.2 ($r = 0.25$, $\mathbf{c} = [0.5\ 0.5]^T$). The pivot point is marked with a circle. Fig.2 shows the set of transform points $\mathbf{q}_i$ and the estimated line δ. The center of gravity $\bar{\mathbf{q}}$ and the point $\mathbf{q}_0$ are marked with a box and a circle respectively. We easily conclude that anyone of the observed points can be selected as the pole of the inversion without affecting the final result.

Example 2. *Noisy data*

When noise is present the selection of the pole becomes critical as different circles, more or less close to the given set of points are determined for different pivot points. Examples of circle fitting for noisy data using the inversion method are shown in Fig.3. There we have three

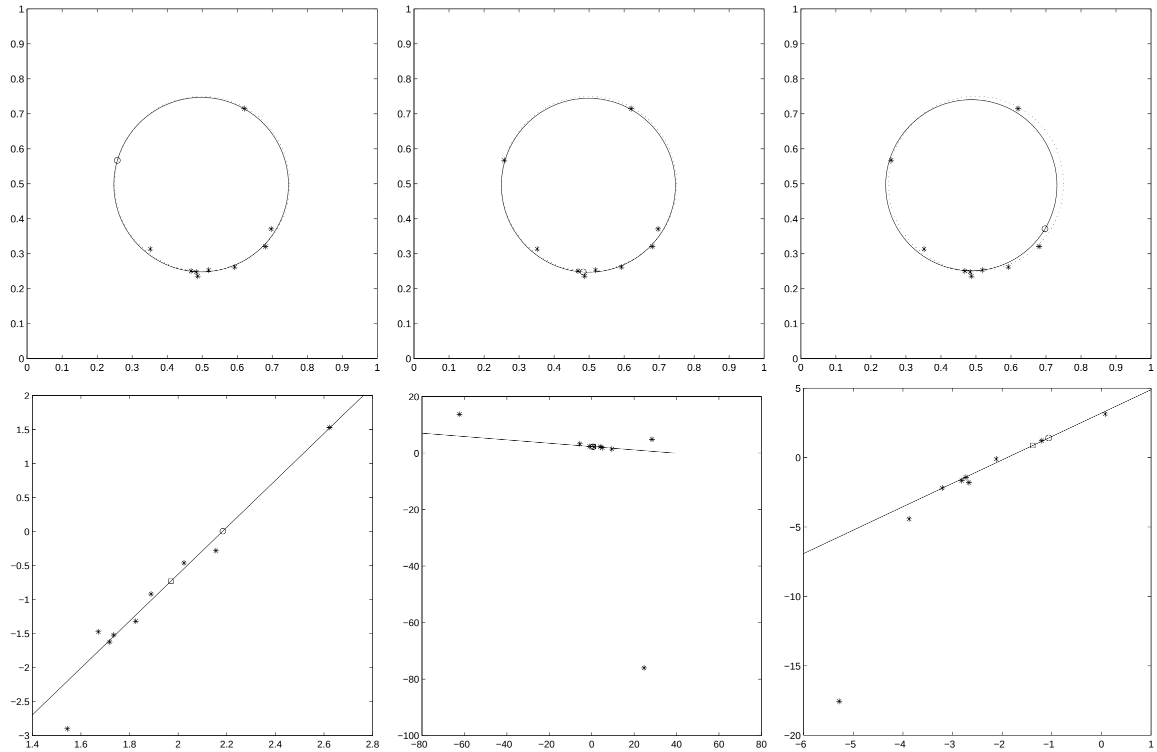


Figure 4: Circle estimation for the same observation points shown in Fig. 3 when the points are weighted based on their distance to the pole.

inversion transformations with different poles, all three using the same coefficient $k = 1$. Top line shows the real circle (dotted line), the estimated one (continuous line), and the given set of points with the selected pole marked by a small circle. The bottom line shows the corresponding transform points $\mathbf{q}_i$, the estimated line δ, the center of gravity $\bar{\mathbf{q}}$ marked by a small square, and the point $\mathbf{q}_0$ marked by a small circle. Clearly the selection of the pole is critical. Distinct circles, more or less close to the given set of points are determined for distinct pole points. Thus the method sketched in [1] should be modified.

Obviously, the errors encountered in estimation of the circle are due to a wrong estimation of the line δ. From the inversion transformation (1) it is immediate that the points close to the pole will be spread in a very wide region after inversion. These are the most sensitive points to any small perturbations from their correct position. A small drift of such a point from the circle is transformed in a very large deviation from the line δ, and hence it has an important contribution to the errors encountered in the estimation of δ. On the contrary, the points situated at a larger distance from the pole are less sensitive to noise. Indeed, their drifts from the correct position will be highly attenuated by the inversion transformation resulting in small deviation from the line δ. Based on these observations it becomes clear that the points has to be treated different according with their distances to the pole. The weight w_i associated with a given observation point p_i has to be a monotonically increasing function of the distance between the point and the pole. In particular, good results have been obtain for $w_i = d_i^4$, where d_i is the distance between the point p_i and the pole.

Example 3. *Noisy data case revisited*

The circles fitted using the proposed method for Example 2 revisited are presented in Fig.4. The algorithm which estimates a circle using the above weighted method is shown in Fig.5. One can note that the estimated circle is quite close to the true circle in all three cases.

So far we have been focused on to the estimation of the fitting circle which passes through one of the observation points. In order to release this constrain, an iterative algorithm has been used. In this case the (first) pole of the inversion can be any point in the plane. The circle estimate is iteratively improved by changing the pole of the transformation at each iteration. The pole used in a certain iteration is the point of the previously estimated circle which is antipodal to the pole used in the previous iteration.

Let $\mathbf{t}_k$ denotes the point selected as pole for the k-th iteration. A circle $C_k(r_k, \mathbf{c}_k)$ which passes through

find circle$(\mathbf{p}_0, \{\mathbf{p}_i \mid i = 1, \ldots, N\})$ {
for $i = 1, \ldots, N$ {
$d_i^2 = (\mathbf{p}_i - \mathbf{p}_0)^T(\mathbf{p}_i - \mathbf{p}_0)$; $w_i = d_i^4$;
$\mathbf{q}_i = \mathbf{p}_0 + \frac{k^2}{d_i^2}(\mathbf{p}_i - \mathbf{p}_0)$;
}
$\bar{\mathbf{q}} = \left(\sum_{i=1}^{N} w_i \mathbf{q}_i\right) / \left(\sum_{i=1}^{N} w_i\right)$;
for $(m, n) \in \{(2,0), (0,2), (1,1)\}$ {
$\mu_{mn} = \left(\sum_{i=1}^{N} w_i (q_i(x) - \bar{q}(x))^m (q_i(y) - \bar{q}(y))^n\right) / \left(\sum_{i=1}^{N} w_i\right)$;
}
compute θ **as in equation** (3);
$d_0 = \mathbf{p}_0^T \mathbf{u}_\theta$;
the radius r **and the center c are given by**
$r = \text{sign}(d_0) k^2 / (2d_0)$;
$\mathbf{c} = \mathbf{p}_0 + \mathbf{u}_\theta k^2/(2d_0)$;
}

Figure 5: The proposed algorithm for estimation of a fitting circle which pass through a given point $\mathbf{p}_0$

$\mathbf{t}_k$ can be estimated using the inversion transformation. The pole of the $k+1$-th iteration is chosen as the point $\mathbf{t}_{k+1}$ of $C_k(r_k, \mathbf{c}_k)$ which is antipodal with respect to $\mathbf{t}_k$. A new circle $C_{k+1}(r_{k+1}, \mathbf{c}_{k+1})$ is thereby estimated using $\mathbf{t}_{k+1}$ as the pole of the inversion. The iteration continues in the same fashion until for a certain K, the distance between $\mathbf{t}_K$ and $\mathbf{t}_{K-2}$ becomes lower than a certain threshold (ϵ).

Example 4. *Circle fitting for an arbitrary first pole of inversion*

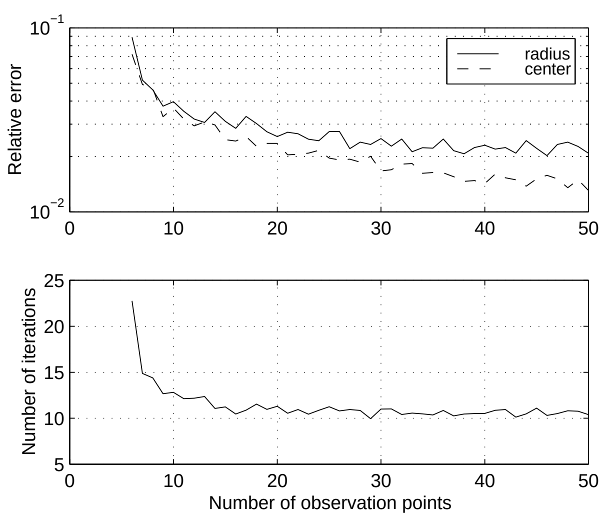


Figure 6: The estimation errors and the number of iterations as functions of the number of observation points.

In this experiment we investigate the ability of our iterative approach to estimate the circle for different number of observation points. The observation points are generated artificially as follows: (i) select a number of points randomly distributed on the circle $C(r = 0.25, \mathbf{c} = [0.5, 0.5]^T)$, (ii) change the position of each observation point by adding a zero mean Gaussian noise of standard deviation 0.025 to the horizontal and vertical coordinates of the point. In each experiment we randomly choose the pole of inversion used in the first iteration as one of the observation points. The algorithm stops at the K-th iteration if the distance between $\mathbf{t}_K$ and $\mathbf{t}_{K-2}$ is lower than 10^{-5}. The estimated circle $C_K(r_K, \mathbf{c}_K)$ is compared against the true circle $C(r, \mathbf{c})$ by computing: (i) the relative error in radius estimation $|r_K - r|/|r|$, and (ii) the relative error in center estimation $\max\{|x_K - x|/|x|, |y_K - y|/|y|\}$. For each number of observation points we performed 100 experiments measuring each time the two estimation errors, as well as the number of iterations performed by the algorithm. The average errors as well as the average number of iterations over these 100 trials are represented as functions of the number of observation points in the Fig.6. The plots show that we can obtain good results for small number of observation points, and the outcomes improve if the number points is increasing.

4 Conclusions

A novel method of circle fitting based on the inversion transformation has been introduced. The novelty consists of (i) weighting the importance of the points according with their distance to the pole of inversion, and (ii) iterating the procedure by choosing at every step a new pole of inversion. Experimental results reveal that our method is capable to determine a quite good approximation of the true circle even for small numbers of observation points, in the presence of noise.

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